Stability analyses of divergence and vorticity damping on gnomonic cubed-sphere grids

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All dynamical cores require diffusion.

Implicit: Embedded in numerical methods

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If C is too small: Energy at the small scales can cause instability

If C is too large: The diffusion causes instability

Contents:

* CAM-FV3: NCAR Community Atmosphere Model (CAM), run with the finite-volume cubed-sphere (FV3) dynamical core.

Explicit diffusion

Artificial diffusion of order $2q$, coefficient ν:

Laplacian ($q = 1$): $\nu\nabla^2\boldsymbol{u}$

Hyperviscosity ($q \geq 2$): $(-1)^{q+1}\nu\nabla^{2q}\boldsymbol{u}$.

Horizontal momentum equations, for $\boldsymbol{u} = (u, v, 0)$:

$$\frac{\partial \boldsymbol{u}}{\partial t} = \cdots + (-1)^{q+1}\nu\nabla^{2\mathrm{q}}\boldsymbol{u}$$

Divergence and vorticity damping

Instead, damp the divergent and rotational modes

$$\frac{\partial \boldsymbol{u}}{\partial t} = \cdots + (-1)^{q+1}\, \nu_D\, \nabla(\nabla^{2(\mathrm{q}-1)} D)$$
$$+(-1)^{q+1}\, \nu_\xi\, \nabla \times (\nabla^{2(\mathrm{q}-1)} \xi \hat{\mathbf{k}}),$$

Divergence: $D = \nabla \cdot \boldsymbol{u}$

Vorticity (relative): $\xi = \hat{\mathbf{k}} \cdot (\nabla \times \boldsymbol{u}), \quad \widehat{\boldsymbol{k}} = (0,0,1)$

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Vorticity (relative): $\xi = \hat{\mathbf{k}} \cdot (\nabla \times \boldsymbol{u})$, $\quad \widehat{\boldsymbol{k}} = (0,0,1)$

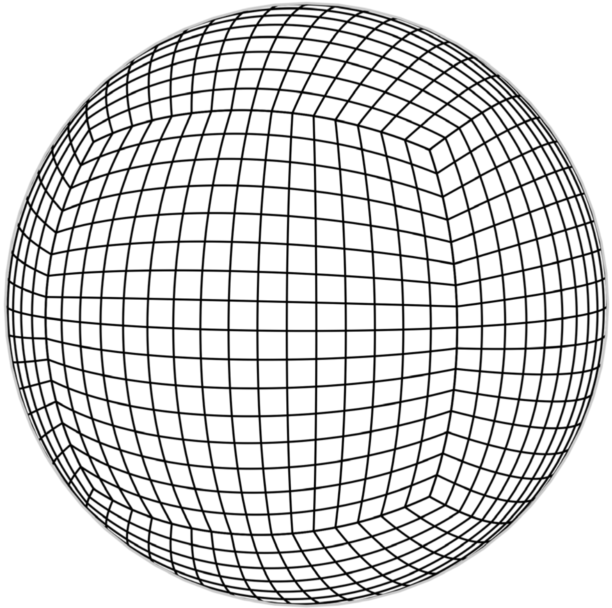
$$\frac{\partial D}{\partial t} = \cdots + (-1)^{q+1}\nu_D \nabla^{2\mathrm{q}} D,$$

$$\frac{\partial \xi}{\partial t} = \cdots + (-1)^{q+1}\nu_\xi \nabla^{2\mathrm{q}} \xi.$$

Gnomonic cubed-sphere grids

Avoids the pole problem of lon-lat grids

Gnomonic: map straight lines from six panels (cube faces) onto great circles on the sphere

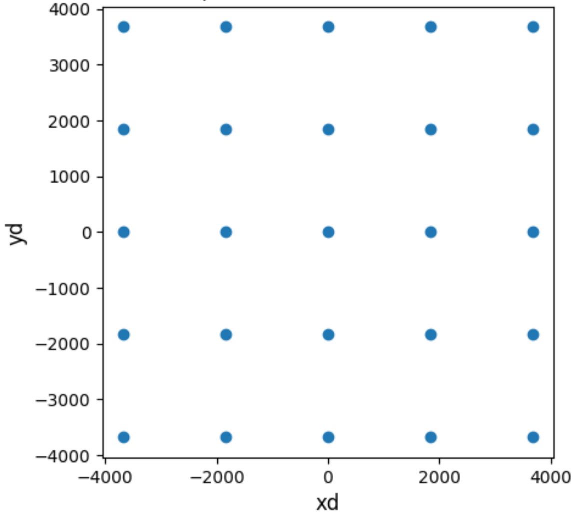


From Anthony Chen

Three cubed-sphere grids

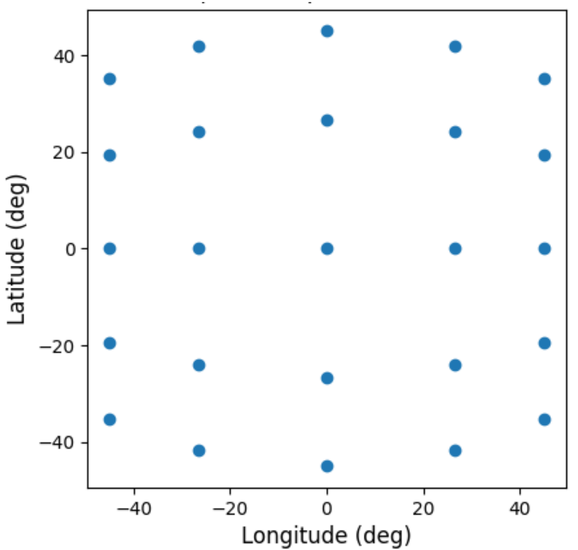
1. **Equidistant**, original grid of Sadournay (1972):
   - Equal spacings on the local panel
     = wide range of cell areas

Local panel

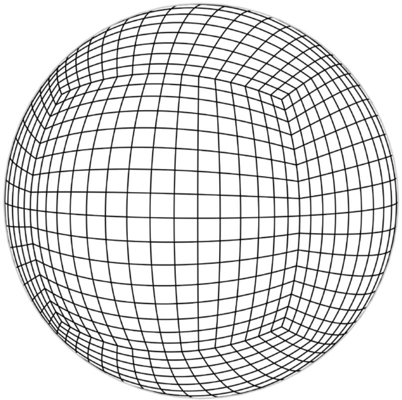


Map

On the sphere



2. **Equiangular** (Ronchi, 1995), most common, used by LFRic (UK Metoffice), SE (NCAR)



3. **Equi-edge**, an additional option (default) for FV3

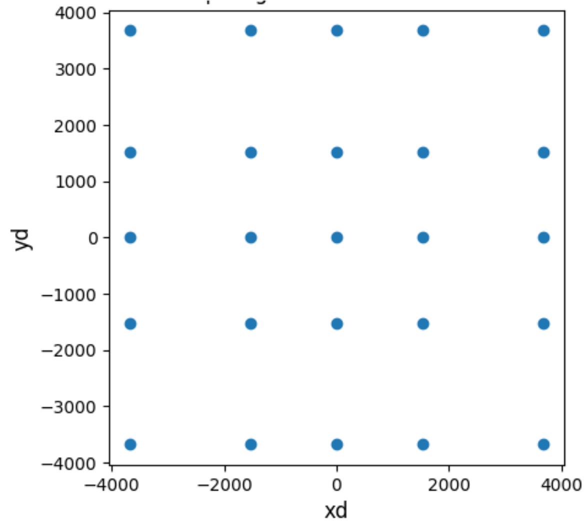
Three cubed-sphere grids

1. **Equidistant**, original grid of Sadournay (1972):

2. **Equiangular** (Ronchi, 1995), most common, used by LFRic (UK Metoffice), SE (NCAR)
   - Equal angles on the sphere = more uniform cells

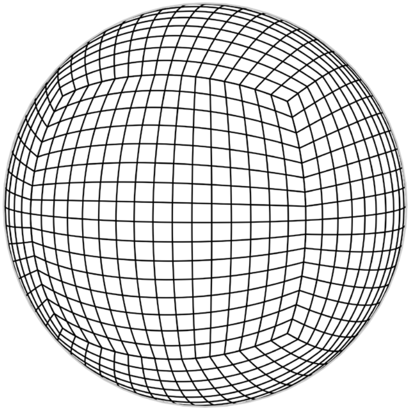
3. **Equi-edge**, an additional option (default) for FV3

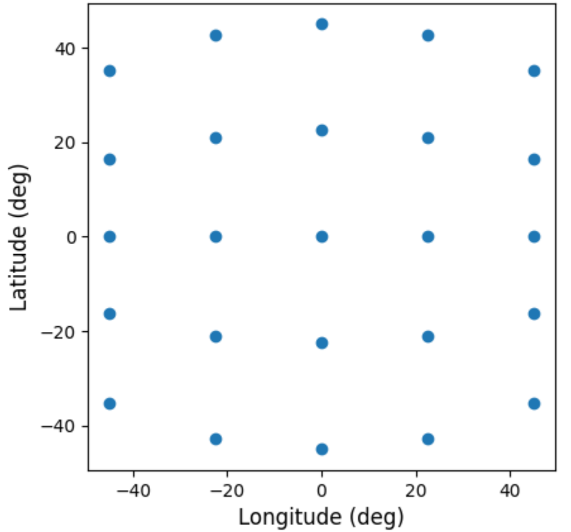
Local panel



Map

On the sphere





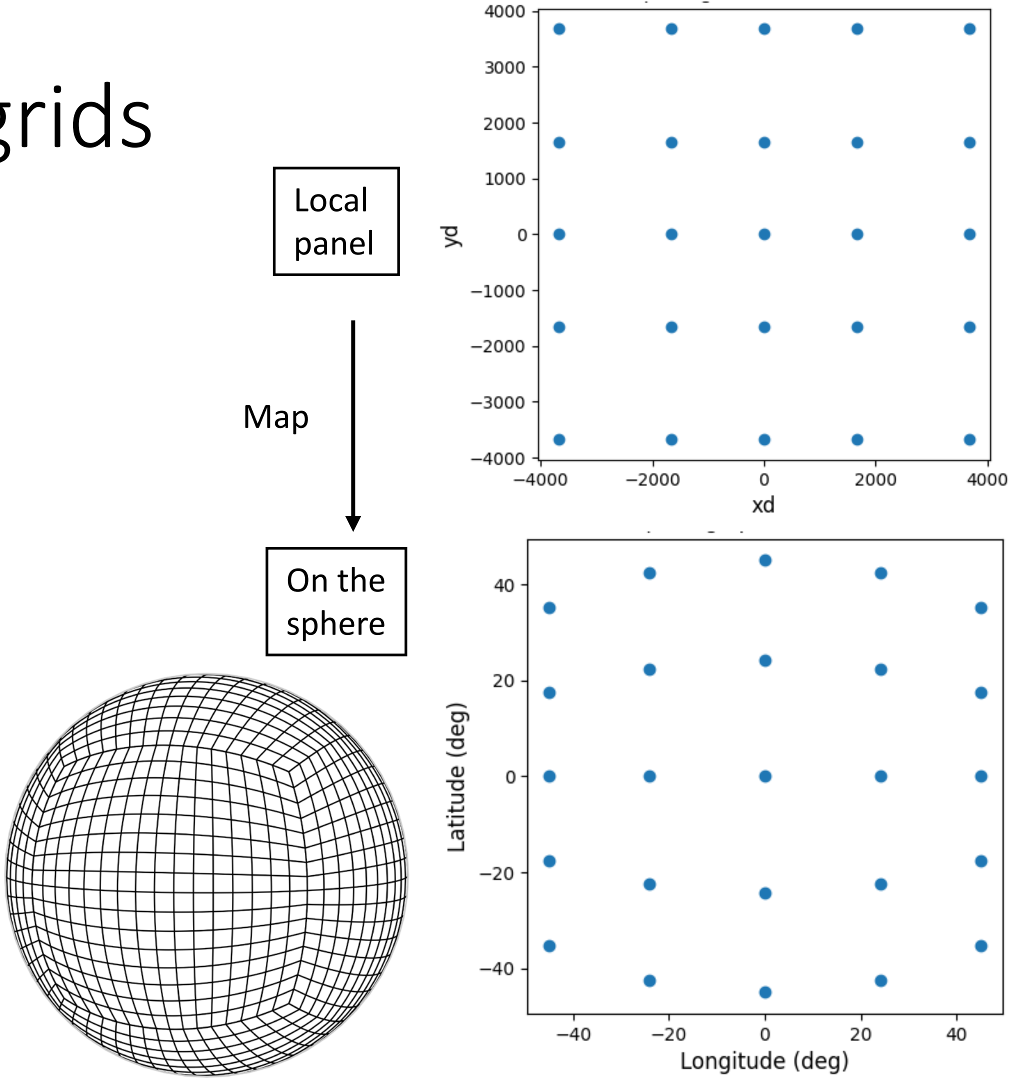
Three cubed-sphere grids

1. **Equidistant**, original grid of Sadournay (1972):
   - Equal spacings on the local panel
     = wide range of cell areas

2. **Equiangular** (Ronchi, 1995), most common, used by LFRic (UK Metoffice), SE (NCAR)

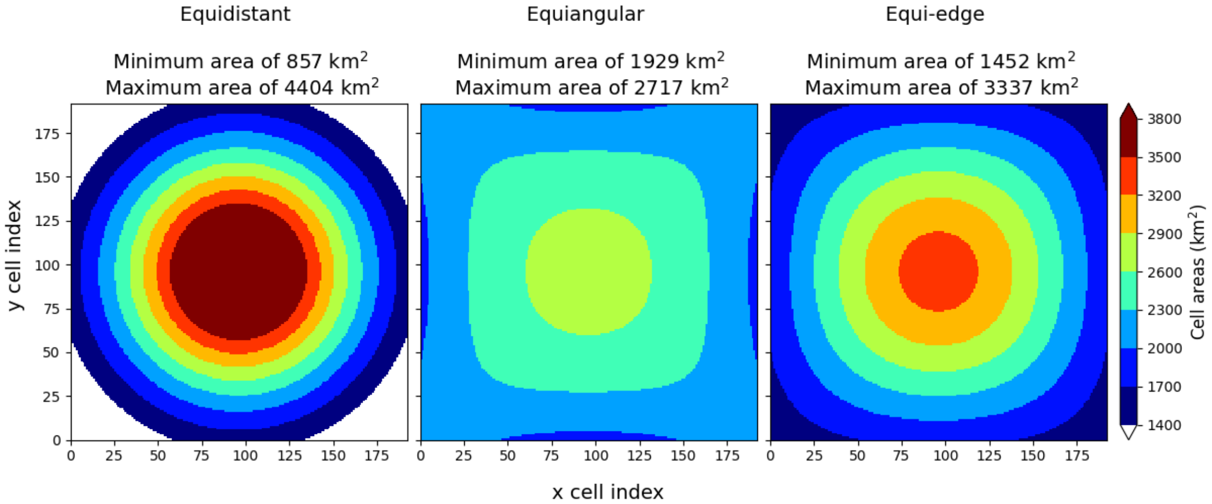
3. **Equi-edge**, an additional option (default) for FV3
   - More uniform cells at panel edges

See Santos (2024) thesis for more details on these grids.



Comparison of cubed-sphere grids:

	Equidistant	Equiangular	Equi-edge
Range of cell areas	Largest	Smallest	
Location of smallest cell	Corners	Middle of edges	Corners
Maximum aspect ratio ($\Delta y/\Delta x$)	1.41 ($\sqrt{2}$)	1.41 ($\sqrt{2}$)	~1.06



Contents:

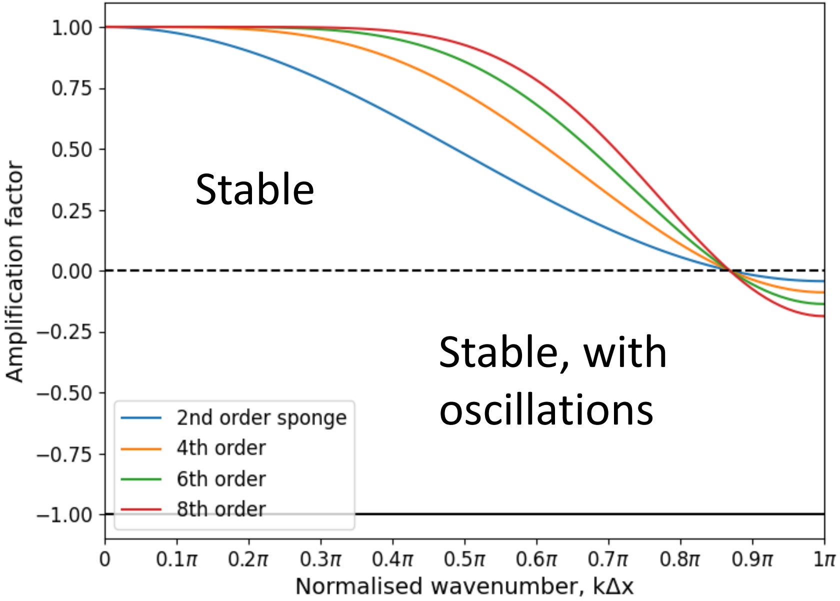
Details of the linear stability analysis are found in ARXIV pre-print: *Stability analyses of divergence and vorticity damping on gnomonic cubed-sphere grids, Andrews and Jablonowski (2025).*

Amplification factors $\boxed{\Gamma = e^{i\omega\Delta t}}$

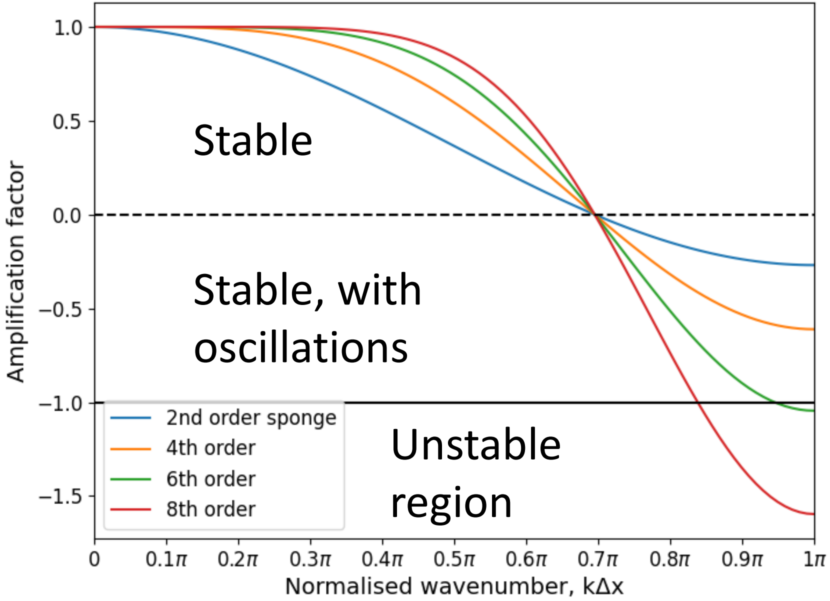
CAM defaults: Laplacian $C_{D,2} = 0.15$ in the upper sponge, hyperviscous $C_{D,2q} = 0.15$

Equi-edge is stable. 6th and 8th order equiangular are **unstable.**

Equi-edge



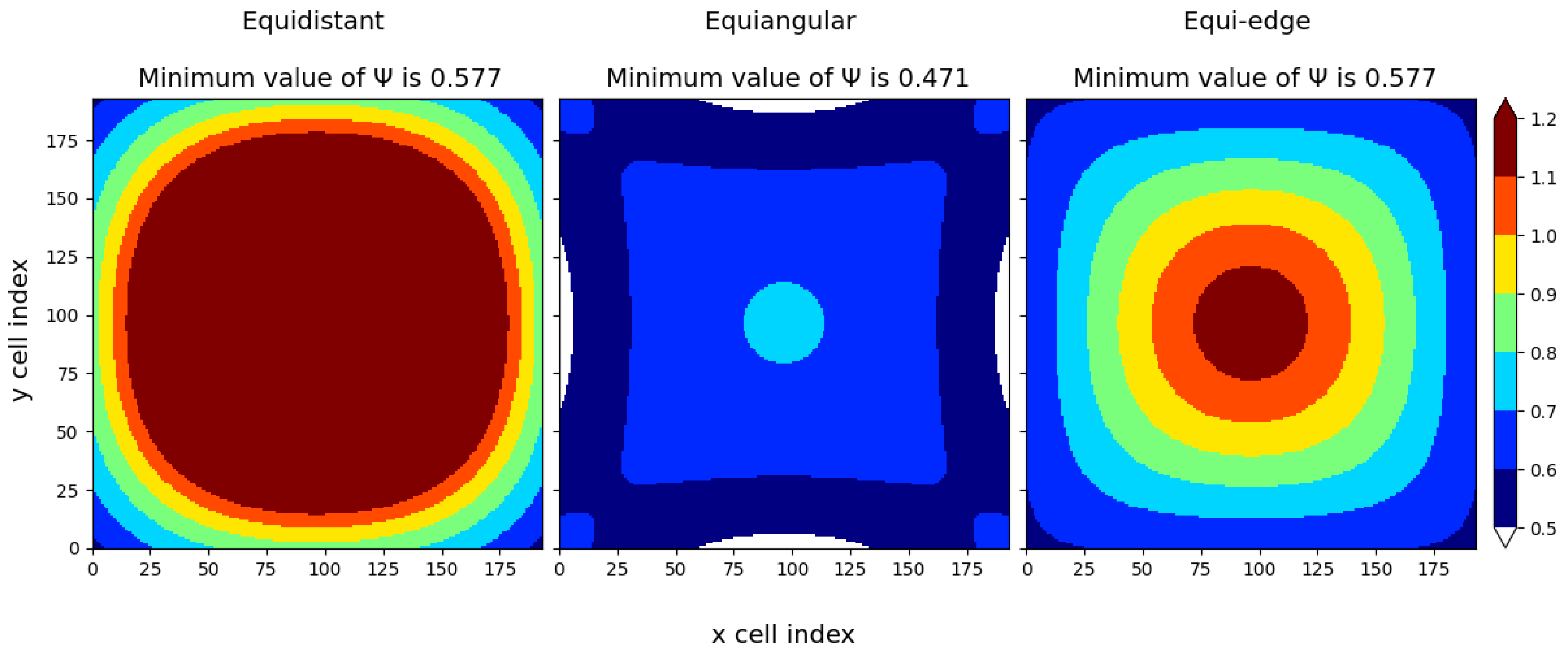
Equiangular



Grid stability function

$$C_{D,2q} \leq 2^{1/q} \frac{\Psi_{c,\min}}{4}$$

Minimum Ψ_c for equi-edge is ~1.22 $\left(\sqrt{2/3}\right)$ greater than equiangular



Corners

Middle of edges

Corners

CAM-FV3 testing

- Use the baroclinic wave test case of Jablonowski, Williamson (2006)
- Run for fifteen days and identify the largest stable $C_{D,2q}$
- Equi-edge and equiangular grids

CAM-FV3* horizontal momentum equations

$$\frac{\partial u}{\partial t} = \left(Y + \mathcal{V}_{y,2q}\right) + \cdots + \delta_x \mathcal{D}_x,$$

$$\frac{\partial v}{\partial t} = -\left(X + \mathcal{V}_{x,2q}\right) + \cdots + \delta_x \mathcal{D}_y,$$

- X, Y transport operators implicitly diffuse ξ but not D
- $\mathcal{D}$ are divergence damping terms (required)
- $\mathcal{V}$ are vorticity damping terms (optional)

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CAM-FV3 divergence damping

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Divergence damping in CAM-FV3

	Equi-edge				Equiangular			
	2nd	4th	6th	8th	2nd	4th	6th	8th
Linear stability limit	0.289	0.204	0.182	0.172	0.236	0.167	0.148	0.140
Default CAM (monotonic), C192	0.291	0.204	0.182	0.172	0.239	0.169	0.151	0.142
Virtually-inviscid unlimited, C192	0.285	0.200	0.180	0.170	0.234	0.167	0.149	None

- Equi-edge can use stronger divergence damping.

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- Equi-edge can use stronger divergence damping.
- Equiangular sixth- and eighth-orders are unstable with CAM default $C_D = 0.15$.

Divergence damping blow-up locations

OMEGA850: Vertical pressure velocity (Pa/s) at 850 hPa.

Using 6th-order divergence damping with $C_{D,6}$ 0.001 above the stable value.

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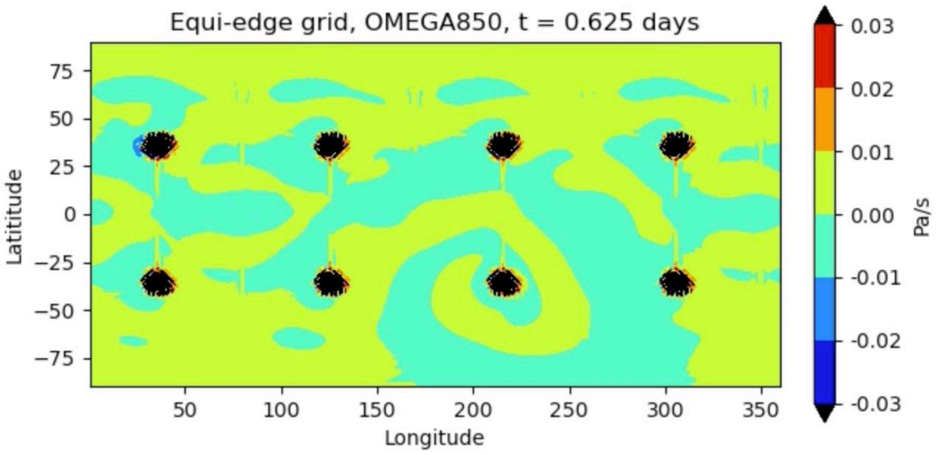

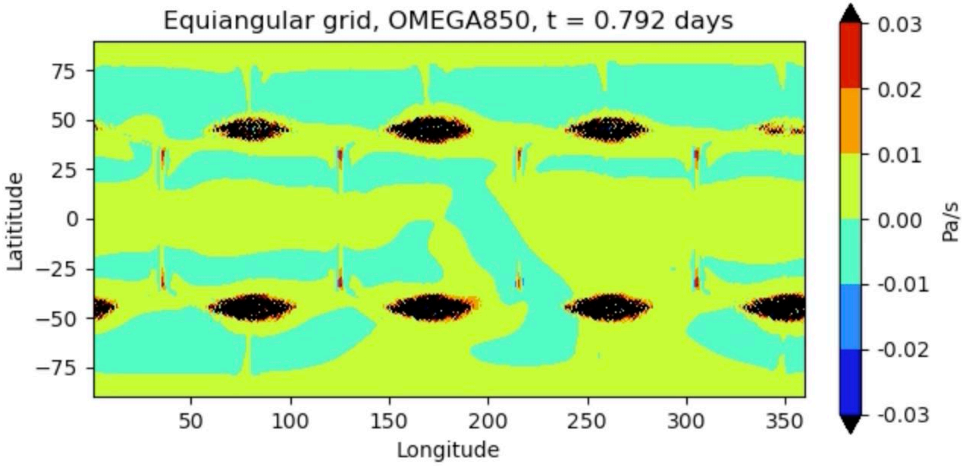

CAM-FV3 vorticity damping

$$\frac{\partial u}{\partial t} = \left(Y + \mathcal{V}_{y,2q}\right) + \cdots + \delta_x \mathcal{D}_x,$$

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Vorticity damping in CAM-FV3

1. Vorticity limits are well below linear theory, due to implicit transport diffusion.

	Equi-edge			
Grid resolution	C96		C192	
Diffusion order	4th	6th	4th	6th
Theoretical	0.203	0.181	0.204	0.181
Default CAM (monotonic)	0.099	0.114	0.090	0.107

Vorticity damping in CAM-FV3

1. Vorticity limits are well below linear theory, due to implicit transport diffusion.

2. Sixth-order has a larger stability range than fourth-order.

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Vorticity damping in CAM-FV3: The transport scheme affects stability

3. Maximum $C_{\xi,2q}$ depends on the horizontal transport scheme.

Monotonic = more constraints, generally more diffusive.

	Equi-edge			
Grid resolution	C96		C192	
Diffusion order	4th	6th	4th	6th
Theoretical	0.203	0.181	0.204	0.181
Lin monotonic	0.097	0.113	0.087	0.104
Intermediate unlimited	0.098	0.113	0.089	0.106
Default CAM (monotonic)	0.099	0.114	0.090	0.107
Virtually-inviscid unlimited	0.104	0.116	0.094	0.108
Huynh monotonic	0.105	0.119	0.098	0.113

Hypothesis: Can this indicate the implicit diffusion from transport?

Key Conclusions

- Equi-edge can use stronger divergence damping than equiangular
  - CAM-FV3 defaults are unstable for equiangular (sixth- and eighth-orders)!

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- Different blow-up locations on the two grids:
  - Equi-edge at panel corners
  - Equiangular at the centre of panel edges

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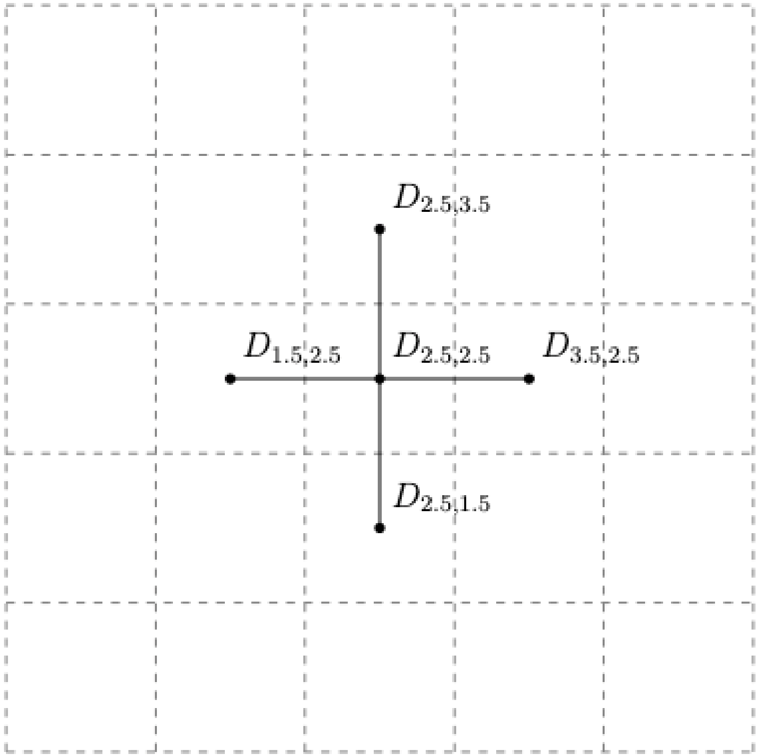
- ARXIV pre-print: *Stability analyses of divergence and vorticity damping on gnomonic cubed-sphere grids, Andrews and Jablonowski (2025).*

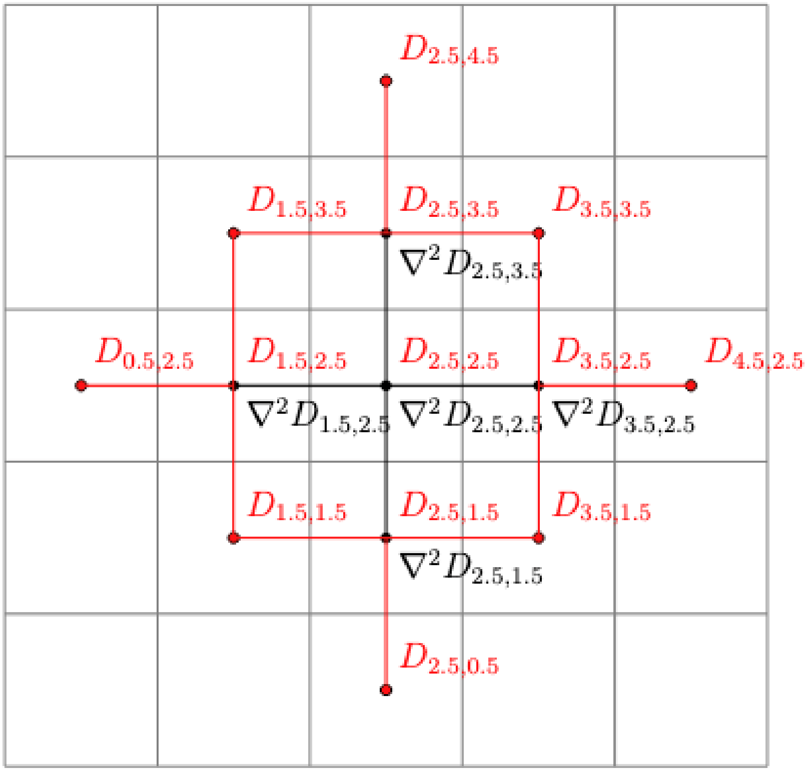
ADDITIONAL SLIDES

Stencils

A higher order of damping requires more ghost cells.



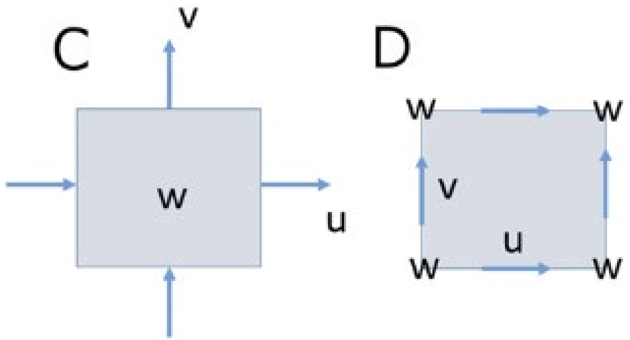
(a) Stencil for Laplacian divergence damping



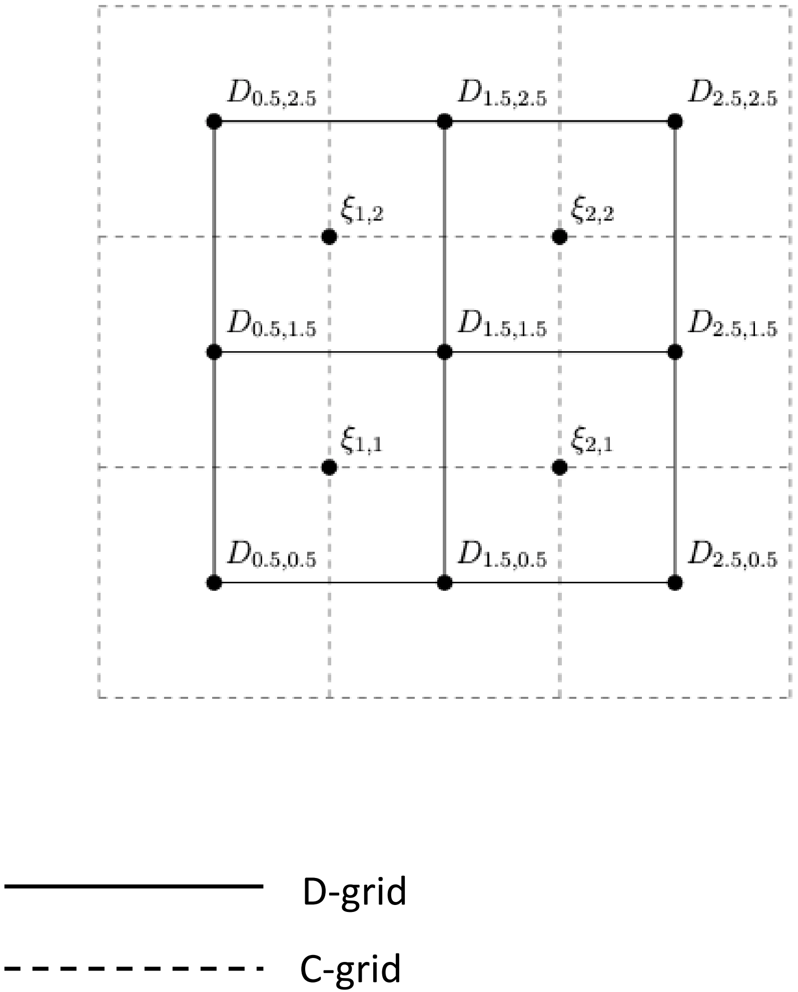
(b) Stencil for fourth-order divergence damping

C- and D-grids

- For application to CAM-FV3, we use the D-grid.
- For the C-grid, the C_D, C_ξ limits are swapped.



Pletzer, Hayek (2018)



Linear stability

- Introduce a grid stability function:

$$\Psi_c(x, y) = \frac{\Delta A_c}{\sin(\alpha)\Delta A_{\min}(\chi + \chi^{-1})}$$

- Then, the linear stability limit is:

$$C_{D,2q} \leq 2^{1/q}\frac{\Psi_{c,\min}}{4}$$

- For vorticity damping, D-grid evaluations are used:

$$C_{\xi,2q} \leq 2^{1/q}\frac{\Psi_{d,\min}}{4}$$

- ΔA_c cell areas on the cubed-sphere
- $\sin(\alpha)$ is internal angle; quantifies non-orthogonality
- ΔA_{min} the smallest cell area
- $\chi = \frac{\Delta y}{\Delta x}$ is the cell aspect ratio

Grid stability function

Minimum value for the equi-edge grid: $\frac{1}{\sqrt{3}} \approx 0.577$

Minimum value for the equiangular grid: $\frac{\sqrt{2}}{3} \approx 0.471$

Ratio is $\sqrt{\frac{2}{3}} \approx 1.22$

There is a larger range of stable coefficients on the equi-edge grid.

$$C_{D,2q} \leq 2^{1/q} \frac{\Psi_{c,\min}}{4}$$

Divergence damping in CAM-FV3

Compare five transport schemes:

Two unlimited, three monotonic.

	Equi-edge				Equiangular			
	2nd	4th	6th	8th	2nd	4th	6th	8th
Linear stability limit	0.289	0.204	0.182	0.172	0.236	0.167	0.148	0.140
Default CAM (monotonic), C96	0.295	0.206	0.184	0.174	0.242	0.171	0.153	0.144
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Virtually-inviscid unlimited, C96	0.296	0.203	0.183	0.173	0.241	0.169	0.151	0.143
Intermediate unlimited, C96	0.291	0.203	0.182	0.173	0.242	0.171	0.152	0.144
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Virtually-inviscid unlimited, C192	0.285	0.200	0.180	0.170	0.234	0.167	0.149	None

Mixed-order divergence damping

$$\frac{\partial D}{\partial t} = \ldots + \nu_{D,2}\nabla^2 D + (-1)^{q+1}\nu_{D,2q}\nabla^{2q} D$$

Fix Laplacian coefficient, $C_{D,2} = 0.05$, then find maximum $C_{D,2q}$

Single-order	Equi-edge				Equiangular			
	2nd	4th	6th	8th	2nd	4th	6th	8th
Linear stability limit	0.289	0.204	0.182	0.172	0.236	0.167	0.148	0.140
Default CAM (monotonic), C96	0.295	0.206	0.184	0.174	0.242	0.171	0.153	0.144

Mixed-order	Equi-edge			Equiangular		
Additional hyperviscosity order	4th	6th	8th	4th	6th	8th
Linear stability limit	0.185	0.171	0.164	0.148	0.137	0.132
Default CAM (monotonic)	0.185	0.171	0.164	0.150	0.139	0.134

	Equi-edge				Equiangular			
Grid resolution	C96		C192		C96		C192	
Diffusion order	4th	6th	4th	6th	4th	6th	4th	6th
Theoretical	0.203	0.181	0.204	0.181	0.167	0.149	0.167	0.149
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Huynh monotonic	0.105	0.119	0.098	0.113	0.110	0.113	0.107	0.111

Computing $\sin(\alpha)$

$$\hat{\mathbf{e}}_{ij} = \frac{\mathbf{p}_i \times \mathbf{p}_j}{||\mathbf{p}_i \times \mathbf{p}_j||}.$$

$$\alpha_{jik} = \arccos(\hat{\mathbf{e}}_{ij} \cdot \hat{\mathbf{e}}_{ik}).$$

$$\alpha = \frac{1}{4}(\alpha_{412} + \alpha_{123} + \alpha_{234} + \alpha_{341}).$$