Why are we still making the Boussinesq approximation in ocean climate models?

(and the implementation of a fully non-Boussinesq option in MOM6)

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MOM6 is available via https://github.com/NOAA-GFDL/MOM6-examples
or at https://github.com/mom-ocean/MOM6

The Boussinesq Approximation

- Non-Boussinesq layered hydrostatic primitive equations:

$$h_k = \rho_k \Delta z_k \text{ (in [kg m}^{-2}\text{])}$$

$$p_{K+½} = p_{Sfc} + g \sum_{k=1}^{K} h_k \qquad \rho_k = \rho(\theta_k, S_k, \overline{p_k})$$

$$\frac{\partial h_k}{\partial t} + \nabla_s \cdot (\vec{u} h_k) = 0$$

$$\frac{\partial \vec{u}_k}{\partial t} + (f + \nabla_s \times \vec{u}_k)\hat{z} \times \vec{u}_k = -\frac{\nabla_s p_k}{\rho_k} - \nabla_s\left(\phi_k + \tfrac{1}{2}\|\vec{u}_k\|^2\right) + \frac{1}{h_k}\Delta\left(\mu \frac{\partial \vec{u}}{\partial z}\right) + \frac{\nabla \cdot \tilde{\tilde{\tau}}_k}{\rho_k}$$

$$\frac{\partial}{\partial t}(h_k \theta_k) + \nabla_s \cdot (\vec{u} h_k \theta_k) = \frac{\Delta Q_k^\theta}{c_p} + \Delta\left(\kappa \frac{\partial \theta}{\partial z}\right) + \nabla_s(h_k K \nabla_s \theta)$$

Boussinesq approximation:

Replace density with a constant (ρ_0) in selected places; often $\rho_0 = 1035$ kg m^{-2}

$$h_k = \Delta z_k \text{ (in [m])}$$ (Following ocean modeling tradition, but we could also have used $h_k = \rho_0 \Delta z_k$)

$$p_{K+½} = p_{Sfc} + g \sum_{k=1}^{K} \rho_k h_k \qquad \rho_k = \rho\left(\theta_k, S_k, p_{Sfc} + g\rho_0(z_{Sfc} - \overline{z_k})\right)$$

$$\frac{\partial h_k}{\partial t} + \nabla_s \cdot (\vec{u} h_k) = 0$$

$$\frac{\partial \vec{u}_k}{\partial t} + (f + \nabla_s \times \vec{u}_k)\hat{z} \times \vec{u}_k = -\frac{\nabla_s p_k}{\rho_0} - \frac{\rho_k}{\rho_0}\nabla_s(\phi_k) - \nabla_s\left(\tfrac{1}{2}\|\vec{u}_k\|^2\right) + \frac{1}{h_k}\Delta\left(\frac{\mu}{\rho_0} \frac{\partial \vec{u}}{\partial z}\right) + \frac{\nabla \cdot \tilde{\tilde{\tau}}_k}{\rho_0}$$

$$\frac{\partial}{\partial t}(h_k \theta_k) + \nabla_s \cdot (\vec{u} h_k \theta_k) = \frac{\Delta Q_k^\theta}{\rho_0 c_p} + \Delta\left(\frac{\kappa}{\rho_0} \frac{\partial \theta}{\partial z}\right) + \nabla_s(h_k K \nabla_s \theta)$$

Historical Advantages of the Boussinesq approximation

- Model can be written in Z-coordinates, not pressure
  - Simple, time-invariant bottom boundary condition
- Model conserves volume, so simulated sea surface never changes if virtual salt fluxes replace fresh water fluxes
  - Global-mean thermosteric sea level changes can be inferred to leading order in post-processing
- Works well with a rigid-lid approximation of the ocean surface
  - Simpler top boundary condition
  - Explicit free surface models replaced rigid lid models with elliptic solvers in the 1990s
- Some divisions by varying density can be avoided for efficiency
  - Instead multiply by a precalculated inverse of the reference density, perhaps incorporated into variable definitions.
  - Even that can be avoided if working in CGS units (as in MOM1-MOM3 or POP) and the Boussinesq reference density is set to 1 g cm^{-3} (e.g., Smith, 2010).
- Convenient simplification for pedagogical purposes

Some Boussinesq ocean models:

MOM1 to MOM5, MOM6 (in GFDL-CM4), POP (in CESM2), MPAS-ocean (in E3SM), NEMO, MITgcm, ROMS (most versions), FVCom, GOLD (in GFDL-ESM2G), …

Adverse Consequences of the Boussinesq approximation

- Model conserves volume, not mass
- Average heat content is volume-weighted, not mass-weighted
- Thermosteric sea-level changes are not directly calculated by the model
  - Global mean Sea Level Rise can be inferred to leading order after the fact
  - Model can not directly simulate sea level rise driven coastline changes
- Wind-driven acceleration errors of up to ~4.5% in fresh coastal waters
- ~1% errors in magnitude of open-ocean diurnal or seasonal temperature cycles
- Vertical dynamic modes have slightly altered structure, which complicates comparisons with observations (e.g., altimetry)

Various published papers document issues with Boussinesq models (e.g, Greatbatch (1994, 2001); Mellor & Ezer (1995); Dukowicz (2001, 2006); McDougall et al (2002); …)

Semi-Boussinesq models relax the Boussinesq approximations in some places (e.g., the dynamic core) but retain it in others (e.g., various parameterizations). Some semi-Boussinesq ocean models:

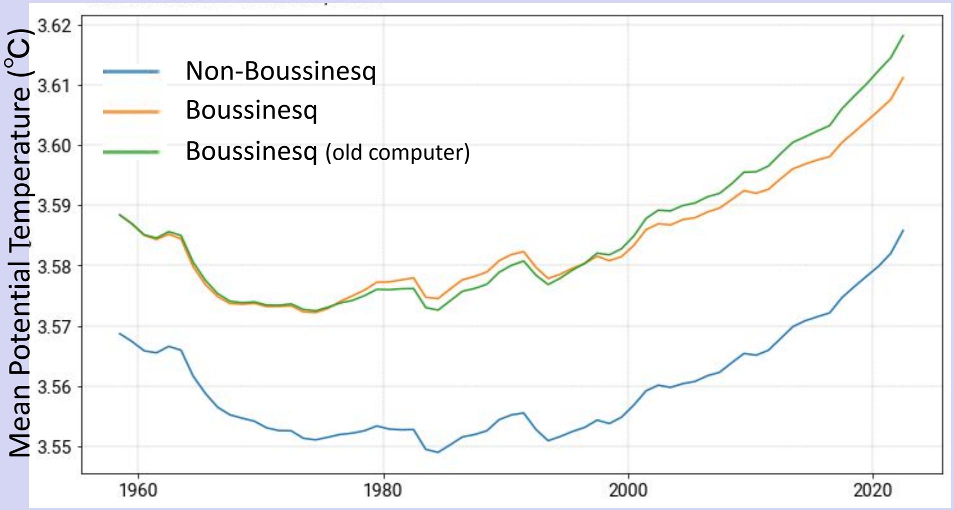
Micom, Hycom, nB-ROMS, POP (some versions), MOM6 (some versions) , …

Fully non-Boussinesq MOM6

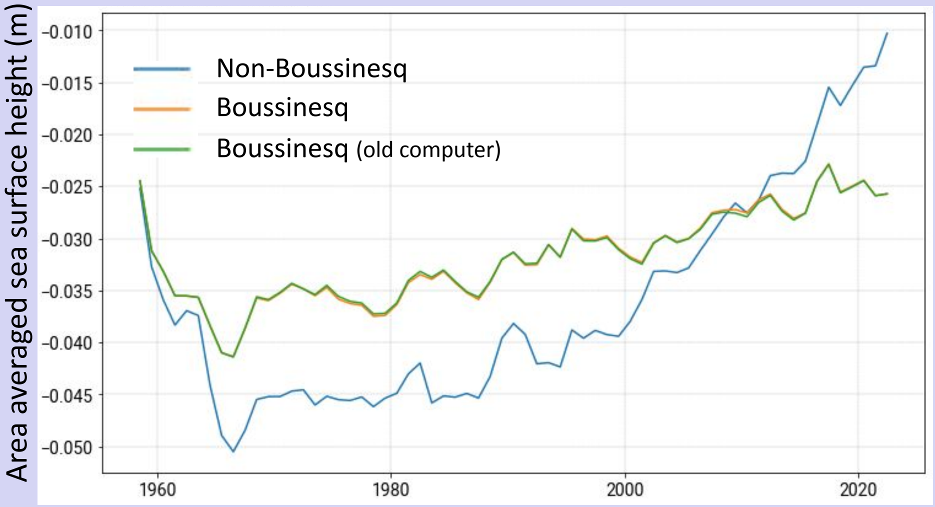
- MOM6 can now be run in fully non-Boussinesq mode or Boussinesq mode
  - Boussinesq mode reproduces existing answers
  - Run-time selection of Boussinesq or non-Boussinesq mode
  - All reinterpretation of input parameters is handled automatically
  - All required code changes are available for use in CESM (some in pending pull requests)
- Testing at GFDL demonstrates non-Boussinesq advantages in global ocean model sea-level simulation, and otherwise similar climates (e.g., SSTs, MLD, currents, AMOC, ENSO, seasonal cycles, …)
- GFDL's OM5 will *very likely* be fully non-Boussinesq

Mean temperature and SSH in JRA-forced ¼° OM5 runs

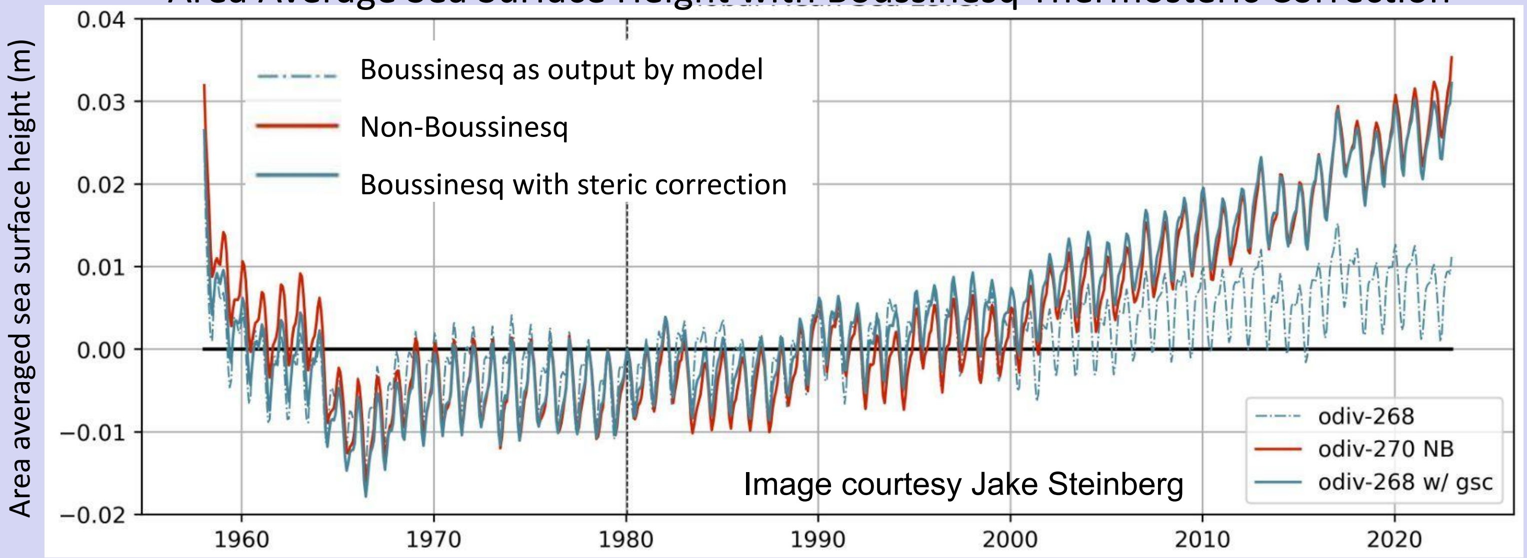
Global Annual Mean Ocean Potential Temperature



Annual-mean Area Average Sea Surface Height



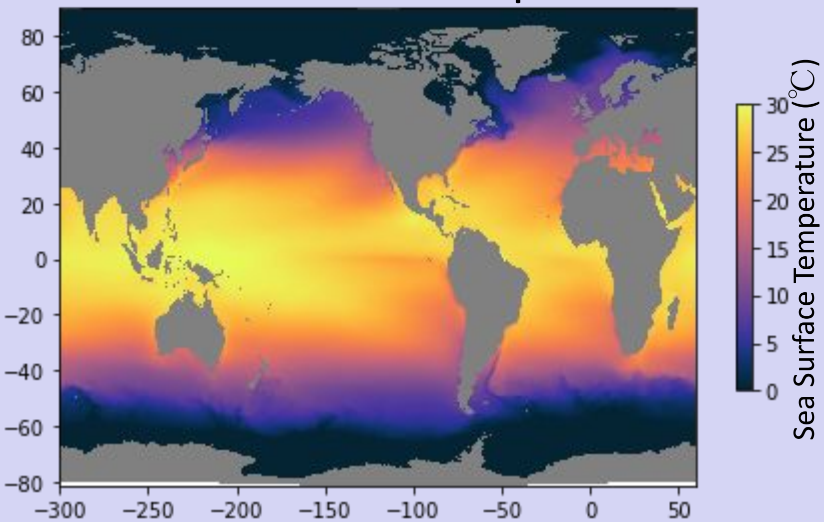
Area Average Sea Surface Height with Boussinesq Thermosteric Correction



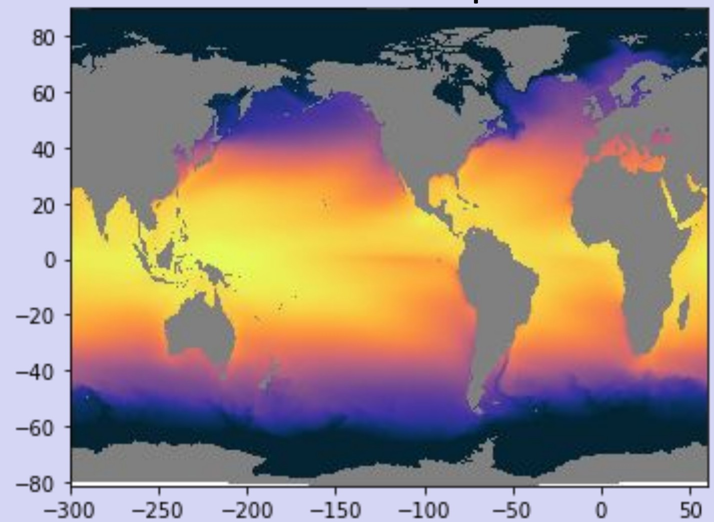
(Please note – this is not a climate projection; there was no spinup and no control run is subtracted off.)

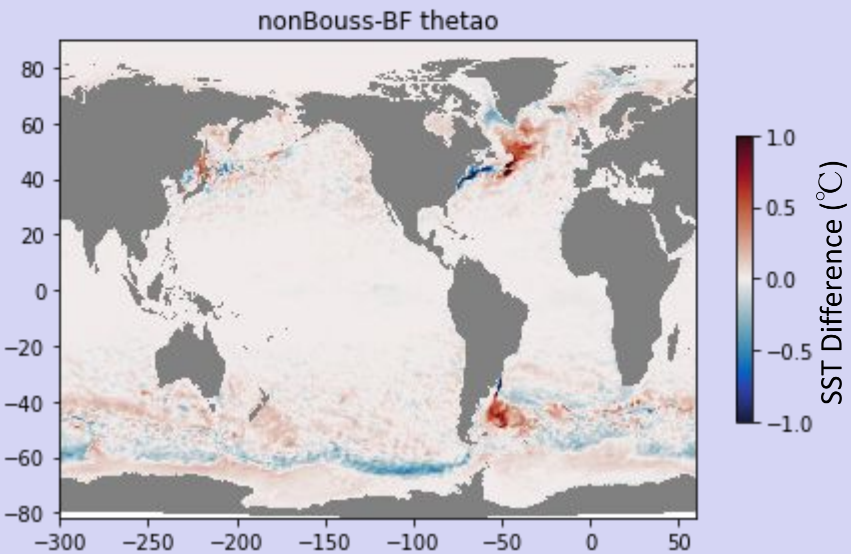
1958-1977 JRA-forced ¼° proto-OM5 SSTs

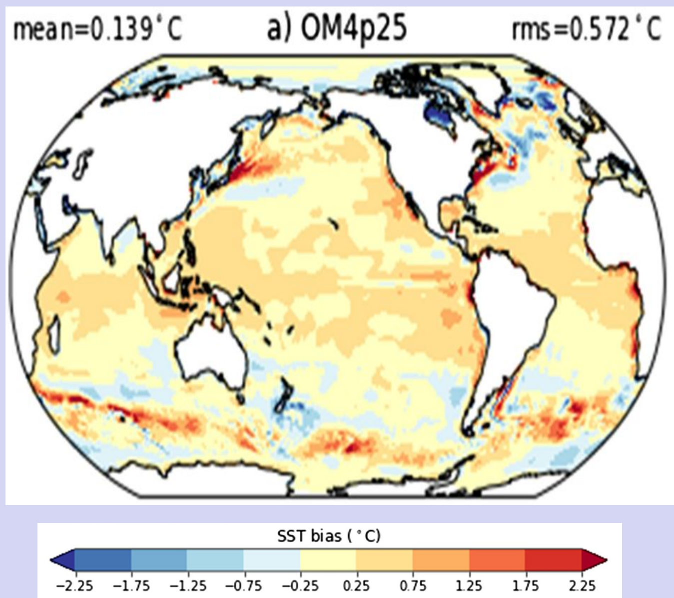
Non-Boussinesq



Boussinesq

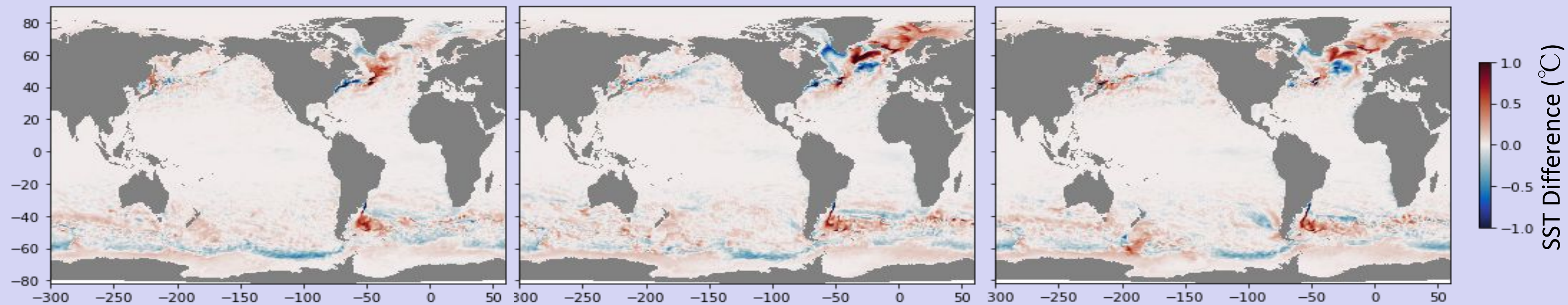




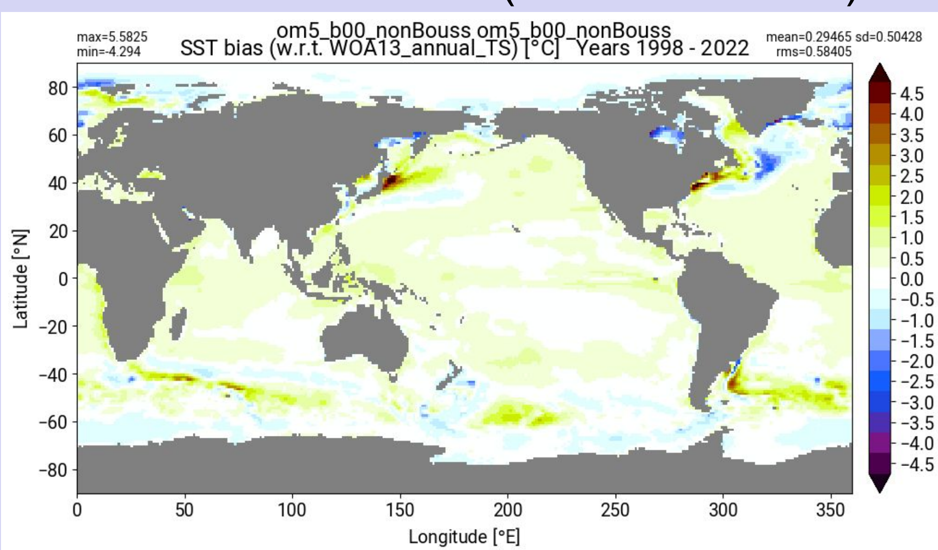


Difference between non-Boussinesq and Boussinesq JRA-forced ¼° OM5 SSTs

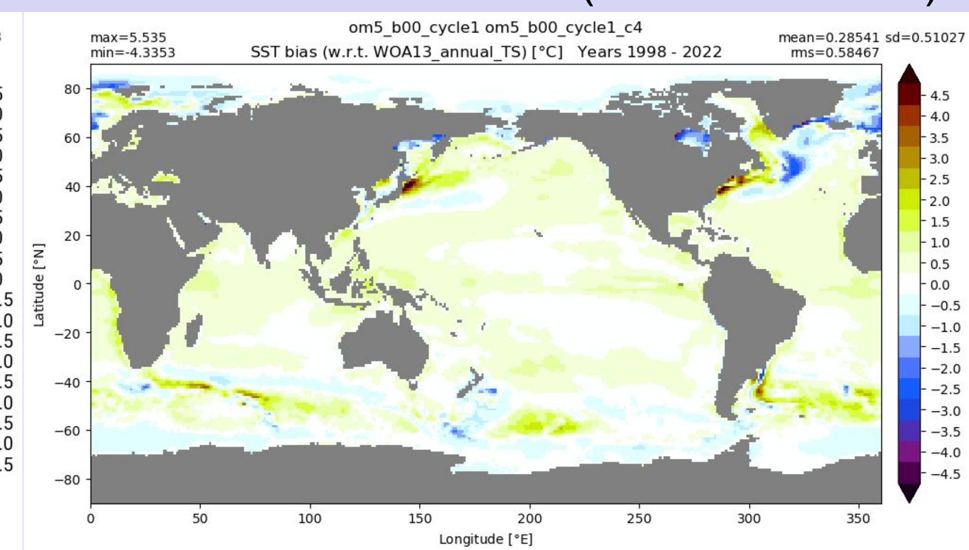
1958-1977 1978-1997 1998-2022



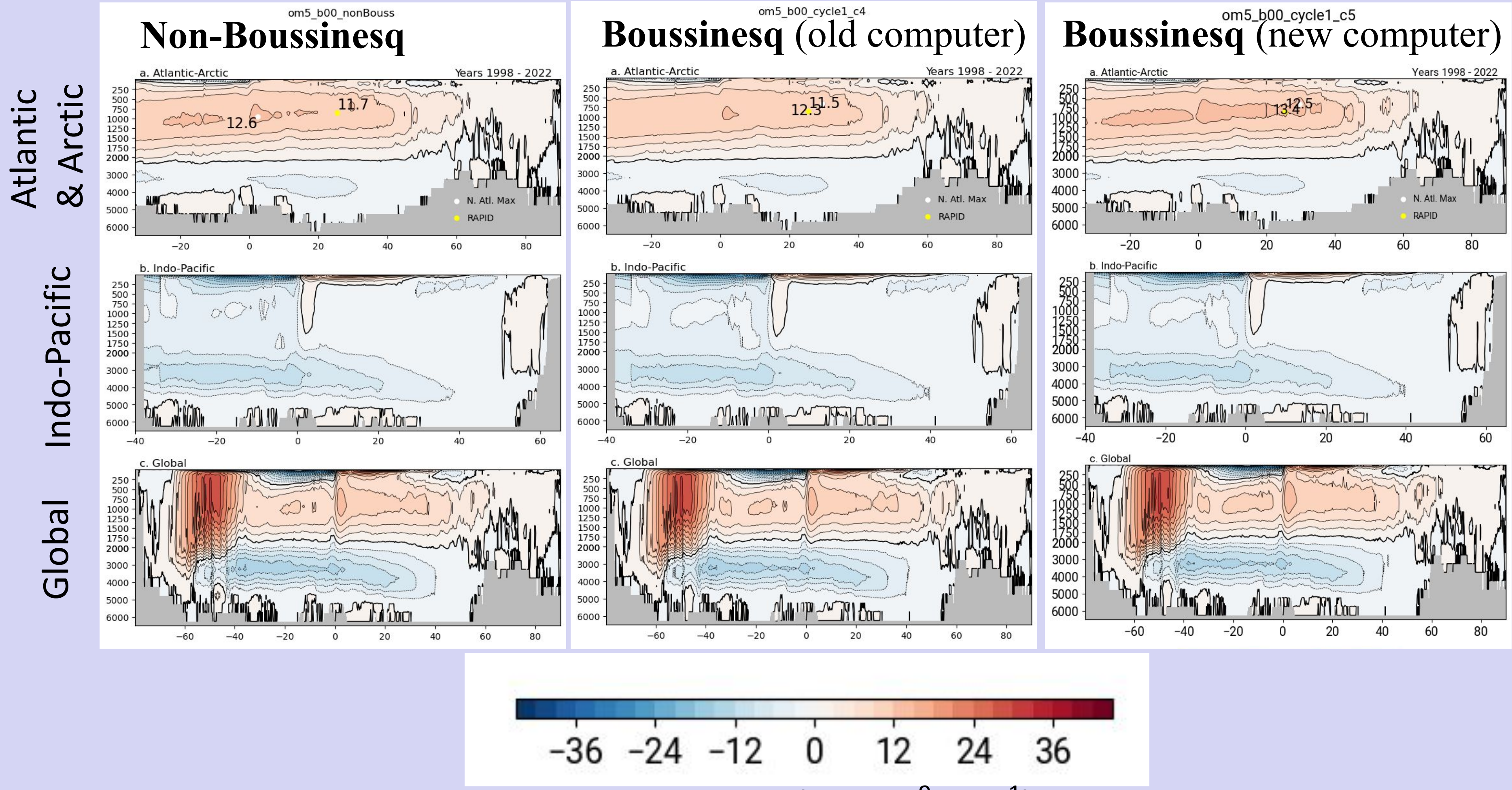
Non-Boussinesq SST Bias vs. WOA13
1998-2022 SST Bias (RMS 0.5841 ℃)



Boussinesq SST Bias vs. WOA13
1998-2022 SST Bias (RMS 0.5847 ℃)



1998-2022 JRA-forced ¼° proto-OM5 Depth-Space Meridional Overturning Circulations

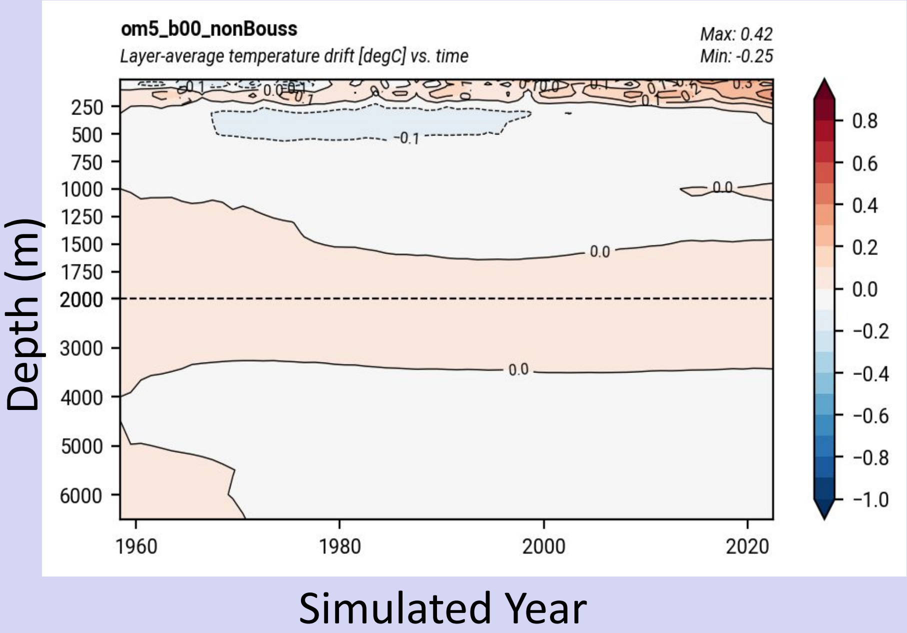


Overturning (Sv = 10^9 kg s^{-1})

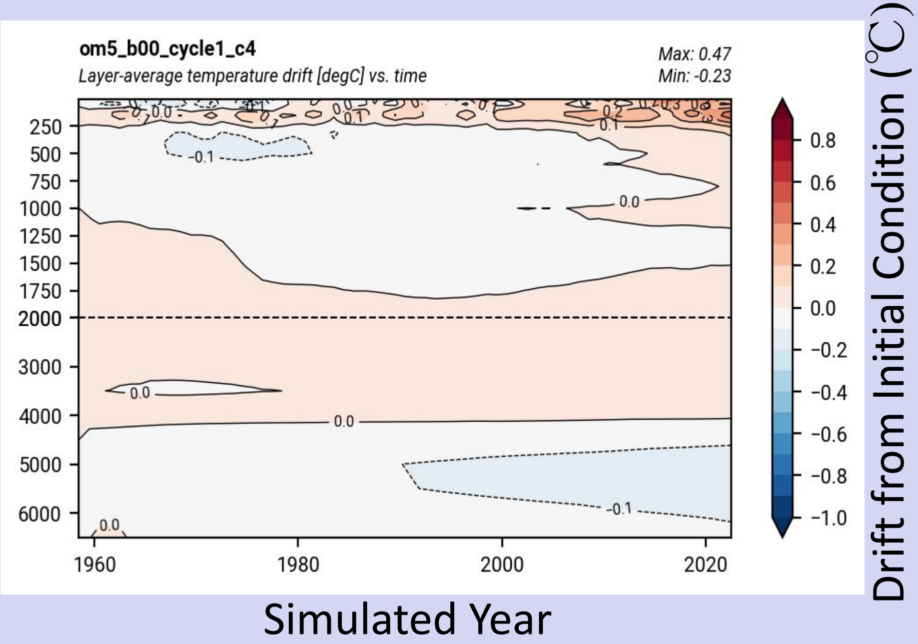
Meridional overturning is very similar between Boussinesq and non-Boussinesq simulations when compared with intrinsic interannual variability.

1998-2022 JRA-forced ¼° proto-OM5 Horizontal-Mean Temperature Drift

Non-Boussinesq



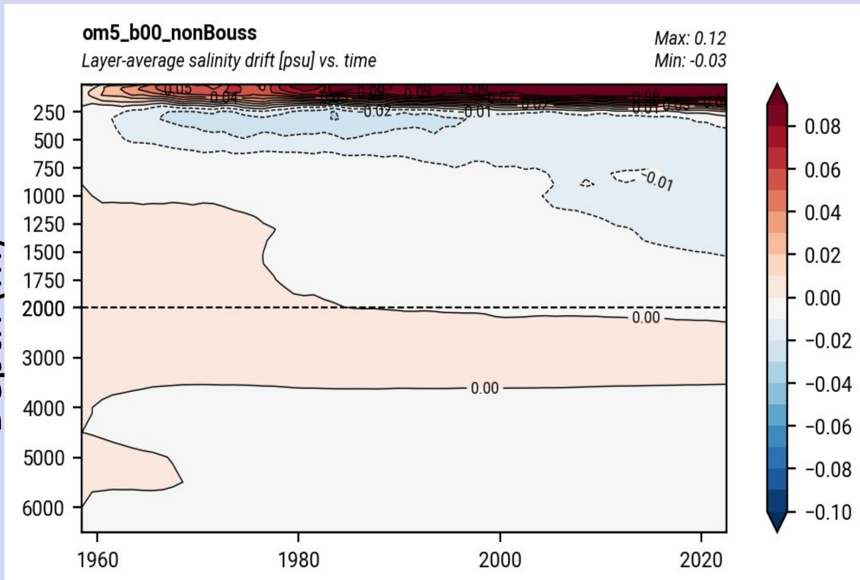
Boussinesq



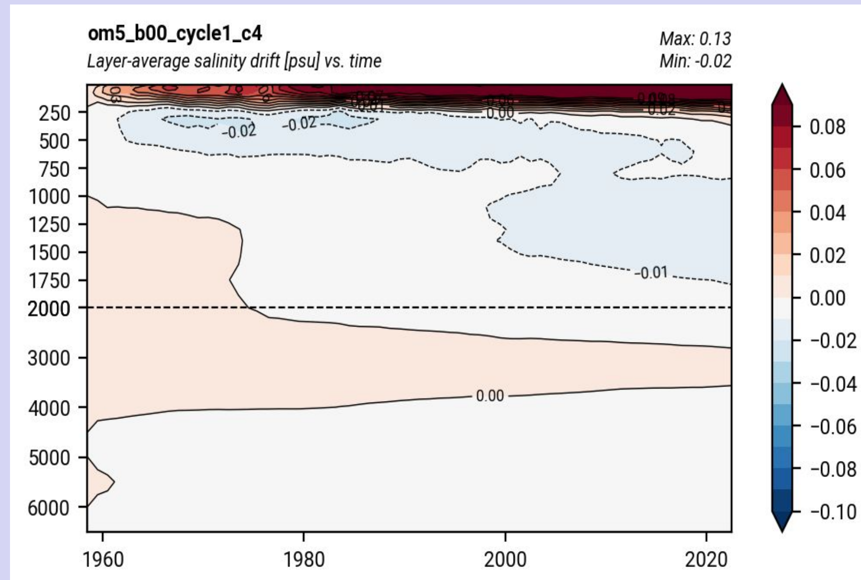
Temperature drift is very similar between Boussinesq and non-Boussinesq runs.

1998-2022 JRA-forced ¼° proto-OM5 Horizontal-Mean Salinity Drift

Non-Boussinesq



Boussinesq



Depth (m)

Simulated Year

Simulated Year

Drift from Initial Condition (ppt)

Salinity drift is very similar between Boussinesq and non-Boussinesq runs.

Converting MOM6 to be fully non-Boussinesq

1. Replace all explicit uses of the Boussinesq reference density with actual density or specific volume
   – Store layer-averaged specific volumes to limit calls to the equation-of-state
2. Replace pressure gradient force calculation (top-down vs. bottom-up)

Pressure Gradient Force with Generalized Coordinates

Hydrostatic, non-Boussinesq pressure with a generalized coordinate (A):

$$\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad \frac{\partial p}{\partial A} = -\rho g \frac{\partial z}{\partial A} \equiv -\rho \frac{\partial \phi}{\partial A} \quad or \quad \frac{\partial \phi}{\partial A} = -\frac{1}{\rho}\frac{\partial p}{\partial A} \equiv -\alpha \frac{\partial p}{\partial A}$$

Hydrostatic pressure gradient force (PGF) in a generalized coordinate (A):

$$\frac{1}{\rho}\nabla_Z p = \frac{1}{\rho}\nabla_A p + \nabla_A \phi = \nabla_A\left(\frac{p}{\rho} + \phi\right) - p\nabla_A \frac{1}{\rho}$$

Non-cancellation truncation errors is problematic when there are two large PGF terms of opposite sign, as occurs with a generalized coordinate!

Different forms are preferred with different specific coordinates:

Pressure-coordinate (non-Boussinesq): $\frac{1}{\rho}\nabla_Z p = \nabla_p \phi$

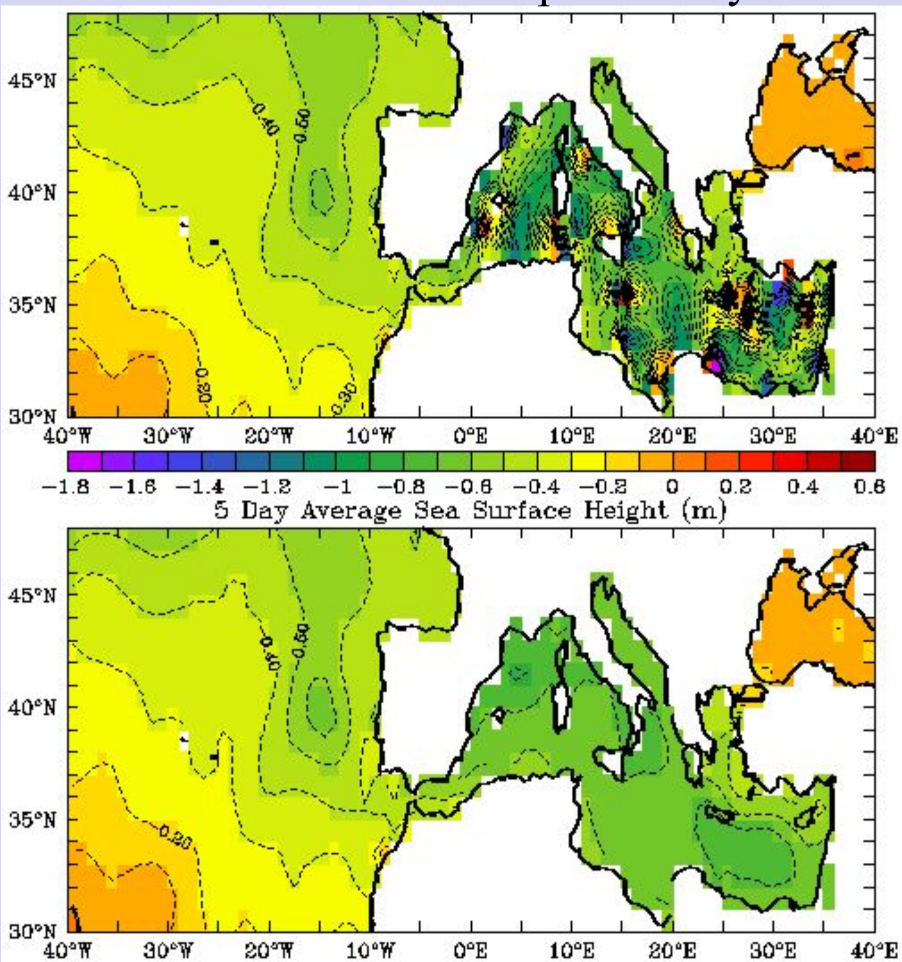
Isopycnal coordinate: $\frac{1}{\rho}\nabla_Z p = \nabla_\rho\left(\frac{p}{\rho} + \phi\right) \equiv \nabla_\rho M$

Z-coordinate (Boussinesq): $\frac{1}{\rho}\nabla_Z p \quad \rightarrow \quad \frac{1}{\rho_0}\nabla_Z p$

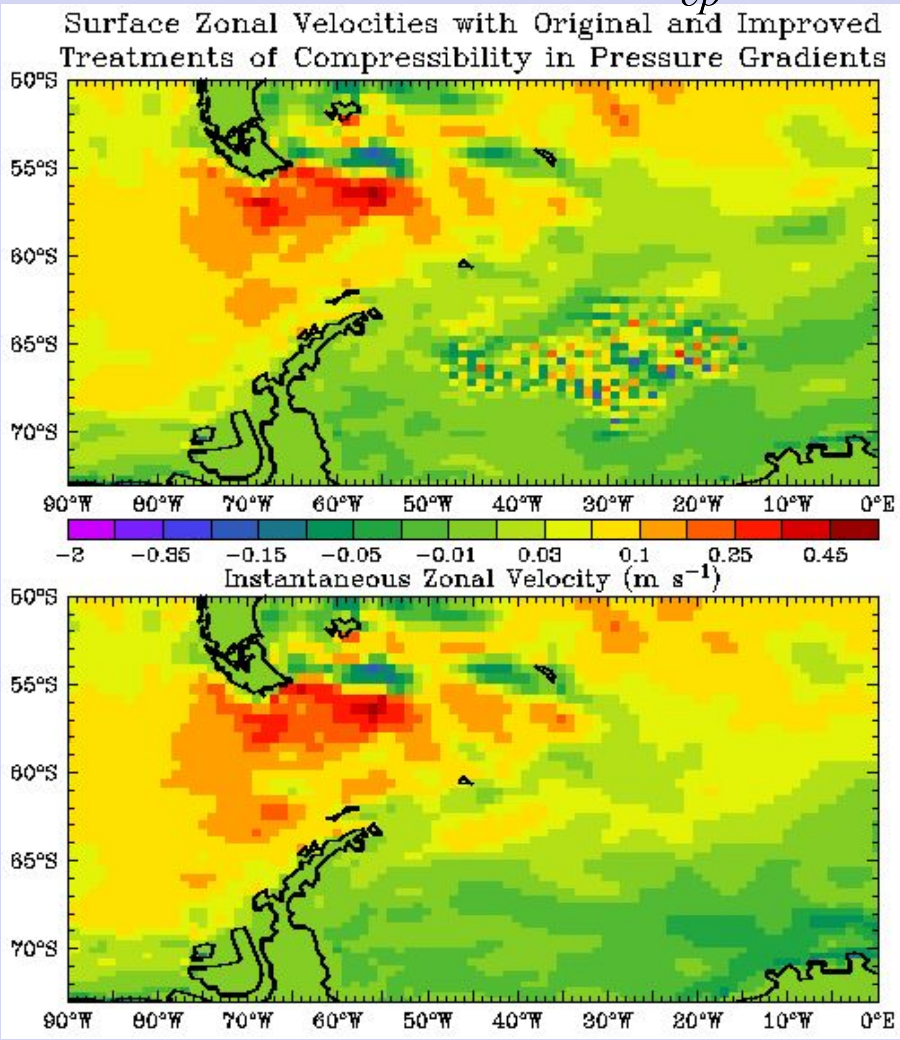
A Thermobaric Instability with a Montgomery Potential PGF

$$\nabla_R(\alpha p+\phi) - p\nabla_R\alpha = \nabla_R\left(\alpha^* p^* + \phi\right) - p^*\nabla_R\alpha^* - \alpha^*\nabla_p p^* \qquad p^* = P(x,y,p) \qquad \alpha^* = \alpha \Big/ \frac{\partial P}{\partial p}$$

Weddell Sea Compressibility



Spatially Varying Fitted Compressibility



Hallberg (Ocean Modelling, 2005)

An analytically integrated finite volume approach using the fully nonlinear EOS avoids these issues without having to determine what compressibility to extract. (Adcroft et al. Ocean Mod., 2008)

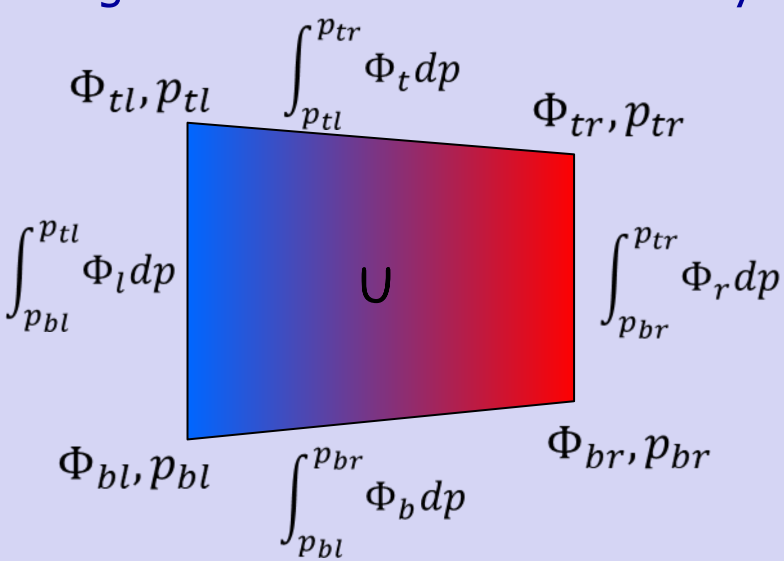
Pressure Gradient Force in MOM6

Integrated finite volume representation of $-\nabla_p \Phi$ (Adcroft et al., Ocean Mod., 2008)

$$\iint_V \frac{\partial u}{\partial t} dx dp = -\iint_V \left.\frac{\partial \Phi}{\partial x}\right|_p dx dp = -\oint_S \Phi dp$$

$$\Phi(p) = \Phi_b - \int_{p_b}^{p} \frac{1}{\rho} dp'$$

With vertically constant, laterally linear θ & S can integrate some forms of EOS analytically

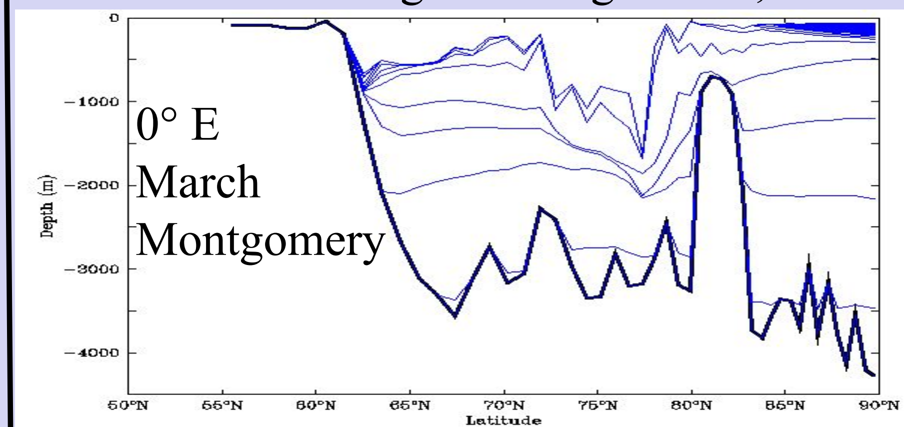


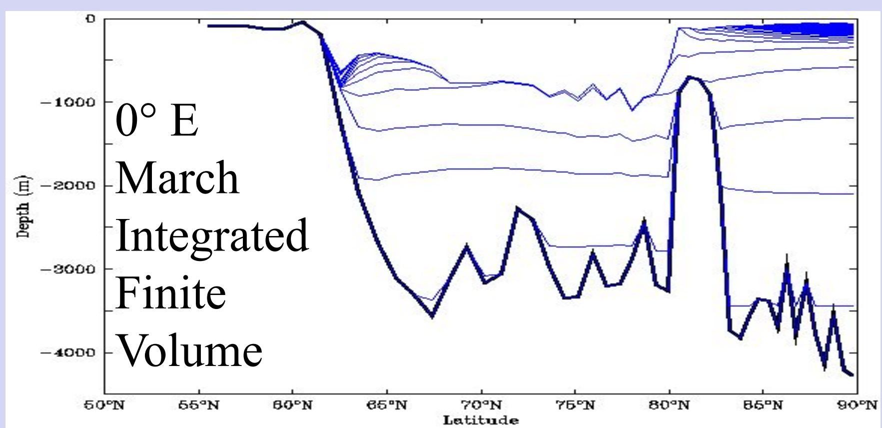
MOM6 often uses Boole's rule quadrature to integrate piecewise linear or parabolic θ & S profiles to find PGF.

$$\frac{1}{\rho}\nabla_z p = \frac{1}{\rho}\nabla_A p + \nabla_A \Phi = \nabla_A\left(\frac{p}{\rho} + \Phi\right) + p\nabla_A \frac{1}{\rho}$$

Thermobaric instabilities emerge in a coupled model with the Montgomery potential form of the pressure gradient force (Hallberg, 2005) .

Transect through Norwegian Sea, Yr 50





Pressure Gradient Force in MOM6

Integrated finite volume representation of $-\nabla_p \Phi$ or $-\nabla_z p/\rho_0$ (Adcroft et al., 2008)

Non-Boussinesq:

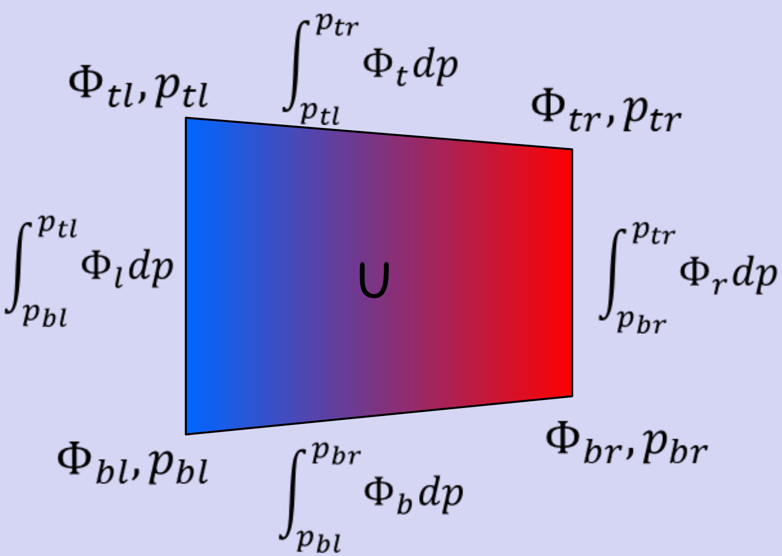
$$\iint_V \frac{\partial u}{\partial t} dx dp = -\iint_V \left.\frac{\partial \Phi}{\partial x}\right|_p dx dp = -\oint_S \Phi dp$$

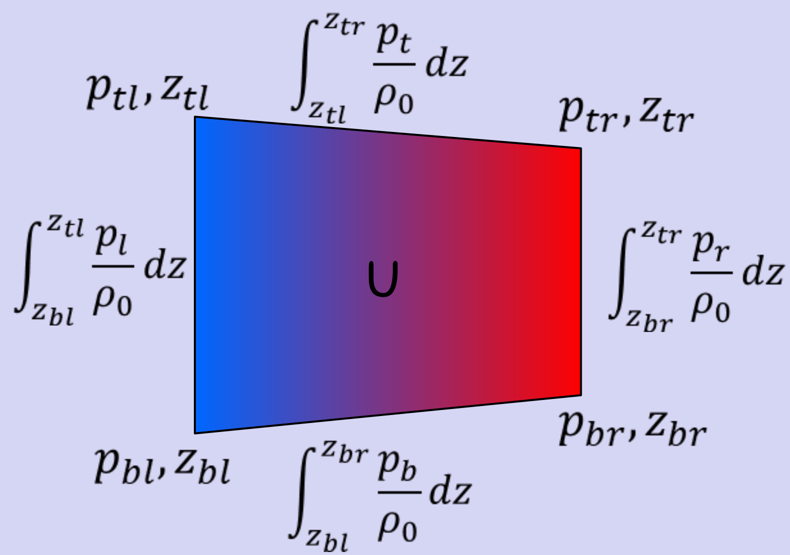
$$\Phi(p) = \Phi_b - \int_{p_b}^{p} \frac{1}{\rho} dp'$$

Boussinesq:

$$\iint_V \frac{\partial u}{\partial t} dx dz = -\iint_V \frac{1}{\rho_0} \left.\frac{\partial p}{\partial x}\right|_z dx dz = -\oint_S \frac{p}{\rho_0} dz$$

$$p(z) = p_t - \int_{z_t}^{z} g \rho dz'$$





With vertically constant, laterally linear θ & S, we can integrate some forms of EoS analytically; MOM6 uses Boole's rule quadrature to integrate other forms of EoS or piecewise linear or parabolic θ & S profiles to find PGF.

Converting MOM6 to be fully non-Boussinesq

1. Replace all explicit uses of the Boussinesq reference density with actual density or specific volume
   – Store layer-averaged specific volumes to limit calls to the equation-of-state
2. Replace pressure gradient force calculation (top-down vs. bottom-up)
3. Reinterpret barotropic solver variables (surface height anomalies vs. bottom pressure anomalies)
4. Use model specific volumes to convert non-Boussinesq thicknesses to heights
   – Done via new functions like `thickness_to_dz()` to hide complexity
5. Verify that changing the Boussinesq reference density does not change non-Boussinesq solutions

Initialization and OBC properties can be set with Z-space inputs

Dimensional Consistency Testing

MOM6 has complete dimensional consistency testing by rescaling 8 units:

1. Time [T ~> s]
2. Density [R ~> kg m^{-3}]
3. Horizontal distance [L ~> m]
4. Vertical height [Z ~> m]
5. Vertical thicknesses [H ~> m] (Boussinesq) or [H ~> kg m^{-2}] (non-Bouss)
6. Heat content (enthalpy) [Q ~> J kg^{-1}]
7. Temperature [C ~> degC]
8. Salinity [S ~> ppt]

- Rescaling each unit by powers of 2 ranging from 2^{-140} to 2^{140} ($\approx 1.4\times10^{42}$) gives bitwise identical answers.
- External packages (e.g., CVMix, TEOS10) are excluded from this testing.
- A “unit scaling type” with conversion factors is passed around the code for conversion to or from mks units for debugging, rescaling constants, etc.

If underflow happens, it has to happen at the same rescaled value.
Rescaling is undone for diagnostics before output.
Reproducing sums via the extended fixed-point have to be unscaled before sums.
Additive adjustment (e.g. changing from °C to °K) leads to changes at roundoff.

Dimensional Consistency Testing

For any choice of integers *T*, *L*, and *Z* all of the following give identical solutions:

Advantages and costs of going non-Boussinesq

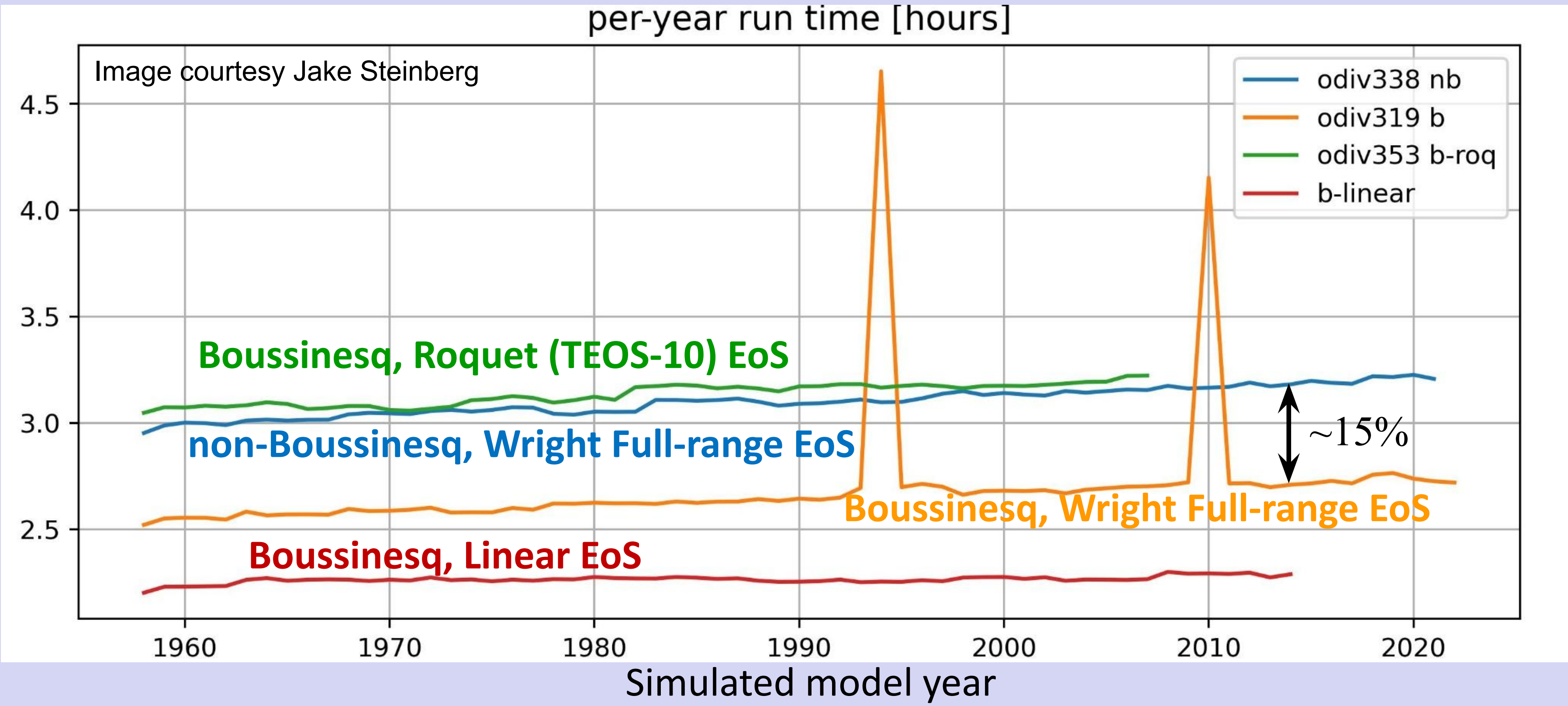
Advantages:

- Mass-weighted conservation properties
- Explicit simulation of steric sea-level changes and related regional patterns; makes moving coastlines a possibility
- Eliminate systematic ~1% to 4% errors in accelerations, etc.
- More direct comparison with some observations

Costs or side-effects:

- ~8.5% larger CPU time in clean tests with OM4_025 global model
  - ~5.1% of this is Pressure gradient force code (~45% slower)
  - Amortized down to a few percent of tracer-heavy Earth System Models?
- Re-interpretation of some input or output variables
  - E.g., transports in [kg s^{-1}] vs. [m^3 s^{-1}]
  - Dynamic viscosities (in [Pa s]) vs. kinematic viscosities (in [m^2 s^{-1}])
- Revised calculation of some diagnostics to replicate Boussinesq counterparts (e.g., mixed layer depths)

JRA-forced ¼° proto-OM5 Ocean/Ice Model Run Times



Note: These runs occurred months apart with different code-versions on a computer with an evolving computational load and decrepit disks, and are informative but not necessarily a clean performance test. The linear trends are probably due to increased iceberg numbers.

A Non-Boussinesq Ocean in CESM?

- Less approximate ocean physics
- Simplified model interpretation and analysis
- All necessary code changes are available now on dev/gfdl branch of MOM6 (PRs still pending for main & dev/ncar)

\- but -

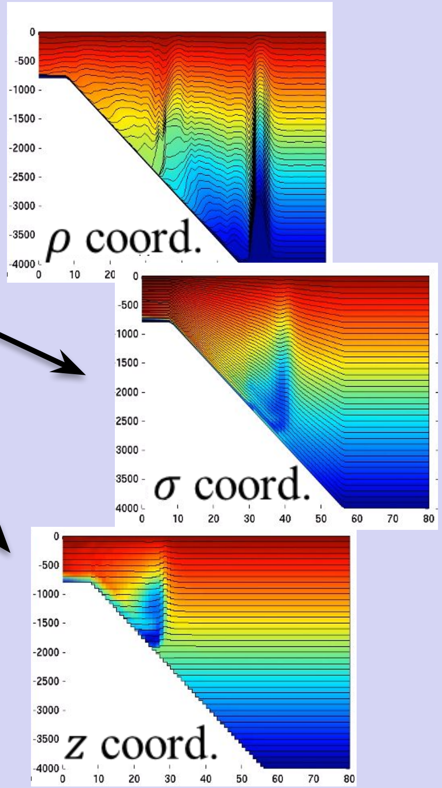
- Slightly more expensive ESMs (< 5%?)
- Some physical model retuning may be advisable
- Ocean BGC tracer physics interfaces are still converted back to be effectively Boussinesq – work might be needed there

The MOM6 community ocean model

Community ocean model rooted in global climate modeling

Vertical Lagrangian Remap Method (VLR)

- General vertical coordinates
- No vertical CFL limit on timesteps or resolution
- Reduced numerical diapycnal mixing for some coordinates
- Structured finite volume for efficiency and conservation
- Efficiencies for biogeochemistry and passive tracers



Novel Capabilities under development

- Coupled ice-sheets with moving calving fronts and grounding lines
- Energetically consistent diapycnal mixing
- Embedded sea-ice and icebergs for stability and verisimilitude
- Capabilities for direct simulation of sea-level and sea-level rise
  - Non-Boussinesq; Wetting and drying (moving coastlines)
  - Tides, including on-line self attraction and loading

Free **Community Open Development** with deliberate ocean model software design…

MOM6 is available via https://github.com/NOAA-GFDL/MOM6-examples
or at https://github.com/mom-ocean/MOM6

The Vertical Lagrangian Remap method

Solve equations in 2 phases:

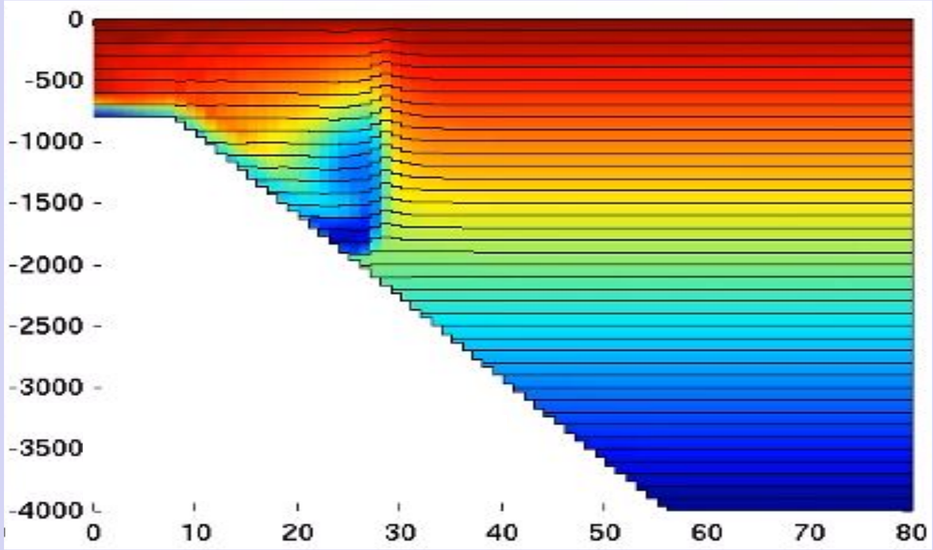
- a **Lagrangian** dynamic update (shallow water eqns.)
- Vertical remapping to an arbitrary (**Eulerian**?) coordinate

Momentum eqn.:

Continuity eqn.:

Tracer eqn.:

Dense-water Overflow Plume in Side-View



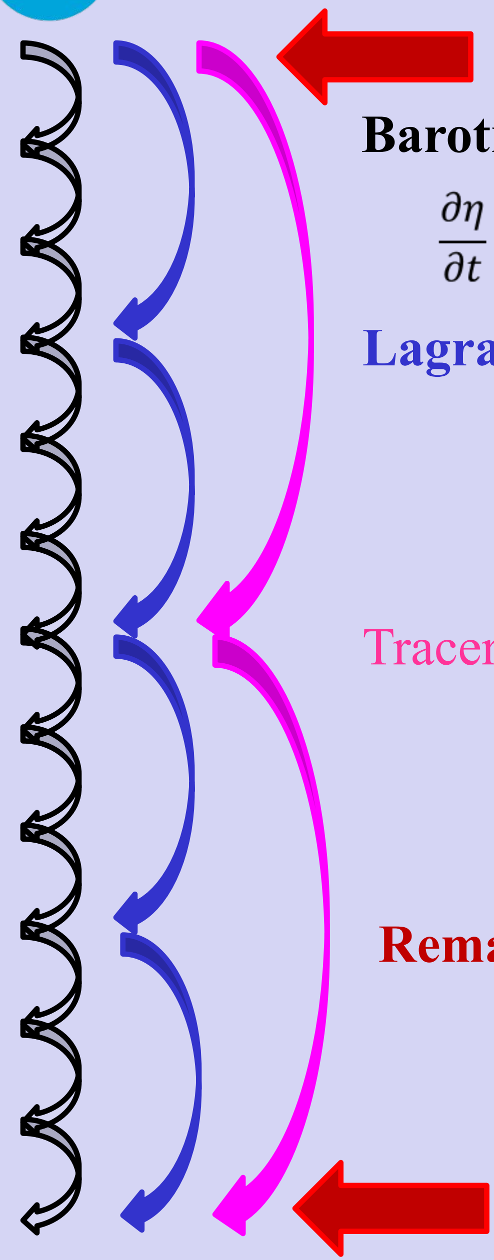
VLR advantages:

- Flexible vertical coordinates
- Remapping imposes no vertical CFL limit on timesteps
- Tracer advection not required to represent gravity waves

See Griffies, Adcroft and Hallberg (*JAMES,* 2020) for a detailed primer on VLR.

4 Time Stepping Cycles in MOM6

(CM4 timesteps)



Barotropic (2-d linear momentum, integrated continuity) **(Δt ~20 *s*)**

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \big((D+\eta)\bar{u}_{BT}\big) = P - E \qquad \frac{\partial \bar{u}_{BT}}{\partial t} = -g\nabla\eta - f\hat{z} \times \bar{u}_{BT} + \bar{F}_{BT}$$

Lagrangian dynamics (3-d Stacked Shallow Water Eqns) **(Δt = 900 s)**

$$\frac{\partial \vec{u}_k}{\partial t} + (f + \nabla_s \times \vec{u}_k)\hat{z} \times \vec{u}_k = -\frac{\nabla_s p_k}{\rho} - \nabla_s\left(\phi_k + \tfrac{1}{2}\|\vec{u}_k\|^2\right) + \frac{\nabla \cdot \tilde{\tilde{\tau}}_k}{\rho}$$

$$\frac{\partial h_k}{\partial t} + \nabla_s \cdot (\vec{u} h_k) = 0$$

Tracer Advection, **Thermodynamics** and Mixing **(Δt = 7200 s)**

$$\frac{\partial h_k}{\partial t} = (P - E)_k$$

$$\frac{\partial}{\partial t}(h_k\theta_k) + \nabla_s \cdot (\vec{u} h_k \theta_k) = Q_k^\theta h_k + \Delta\left(\kappa \frac{\partial \theta}{\partial z}\right) + \nabla_s(h_k K \nabla_s \theta)$$

Remapping and coordinate restoration **(Δt = 7200 s)**

$$h_k^{new} = \Delta_k z_{Coord} \qquad \sum h_k^{new} = \sum h_k^{old}$$

$$\vec{u}_k^{new} = \frac{1}{h_k}\int_{z_{k-½}}^{z_{k+½}+h_k} \vec{u}^{old}(z')dz' \qquad \theta_k^{new} = \frac{1}{h_k}\int_{z_{k-½}}^{z_{k+½}+h_k} \theta(z')dz'$$