Exploring the non-stationarity of coastal sea level probability distributions

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Outline

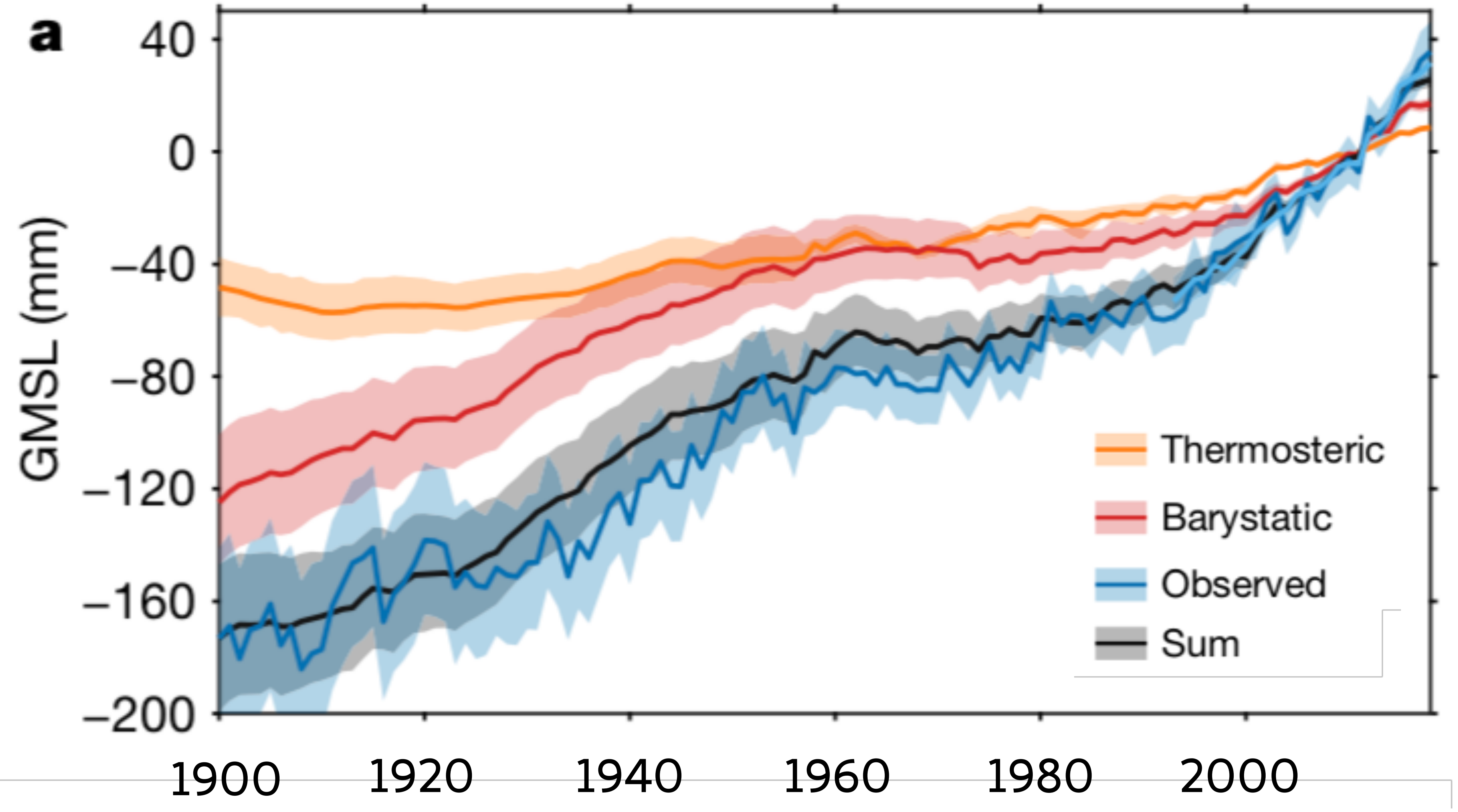
- **Introduction and motivations**
  - Quantifying trends in Global and regional Sea level
  - Climate change (natural or anthropogenic) involves changes in probability distributions
- **Methodology**
  - Quantifying **source** of change in Probability Distributions from time series
- **Results**
  - Changes in coastal sea level Probability Distributions across observation and GFDL model
- **Conclusions**
  - Ongoing and future work

Motivations

Quantifying trends in sea level rise

Global Mean Sea Level

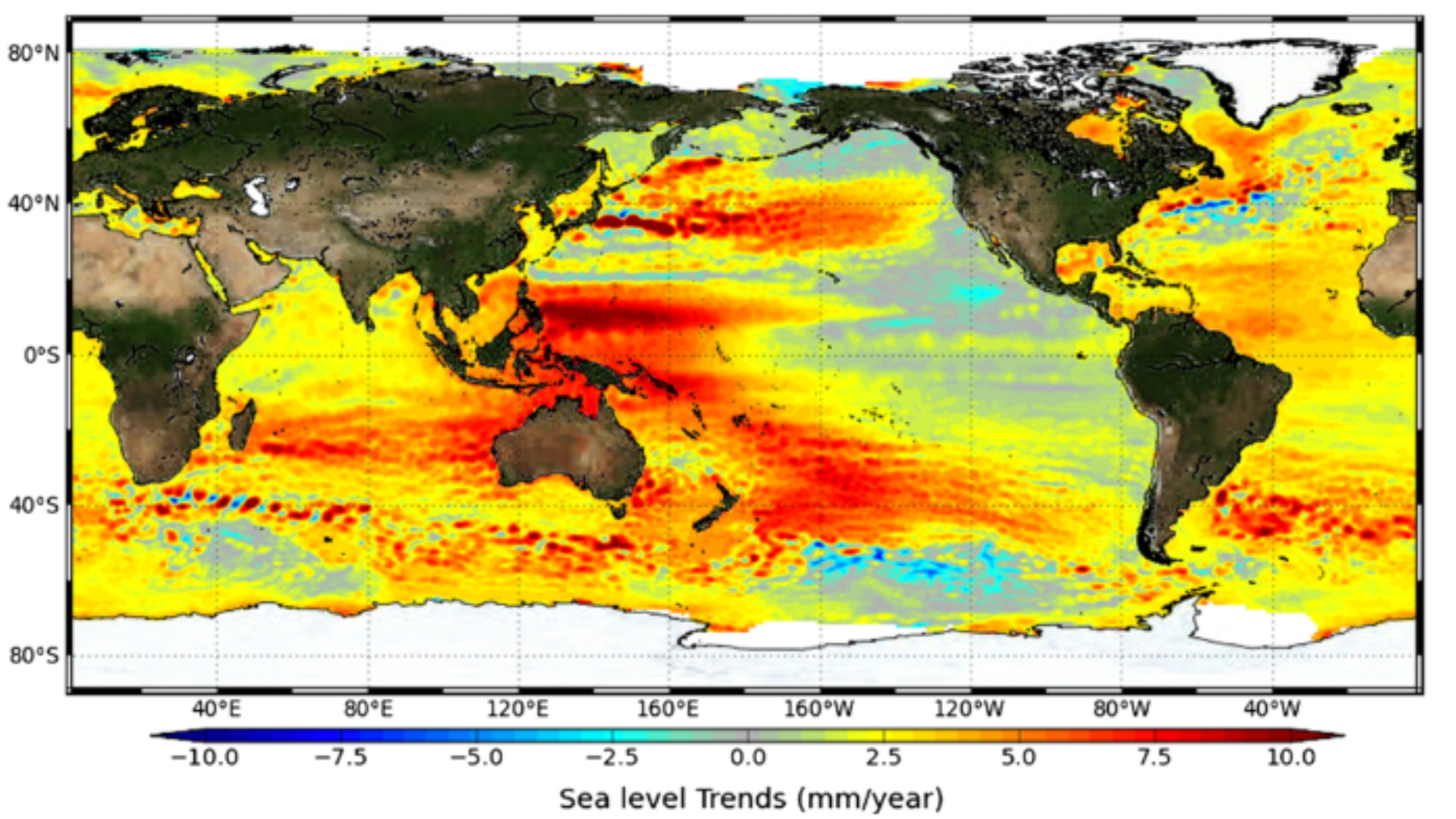
Average rate of 1.35 mm/yr



Frederickse et al. (2020)

Regional Sea Level

Period 1993-2014



Ablain et al. (2020)

Motivations

Quantifying “trends” in sea level time series

- Usually: linear regression
  - It allows to quantify changes in the “mean” of the distribution
  - It ignores higher order changes

- Studies on extremes:
  - Extreme value theory: often assuming that the main changes come from the mean

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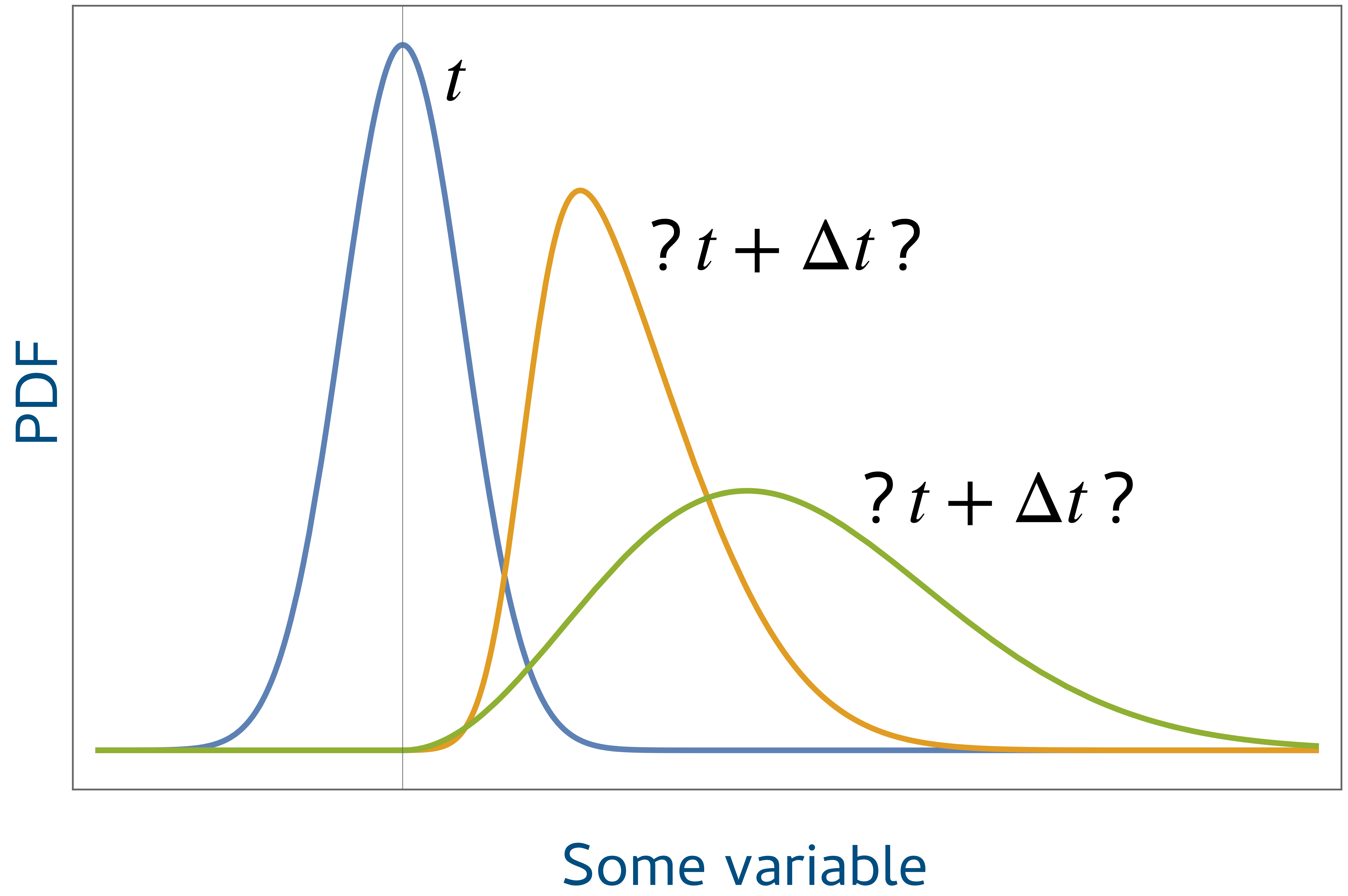
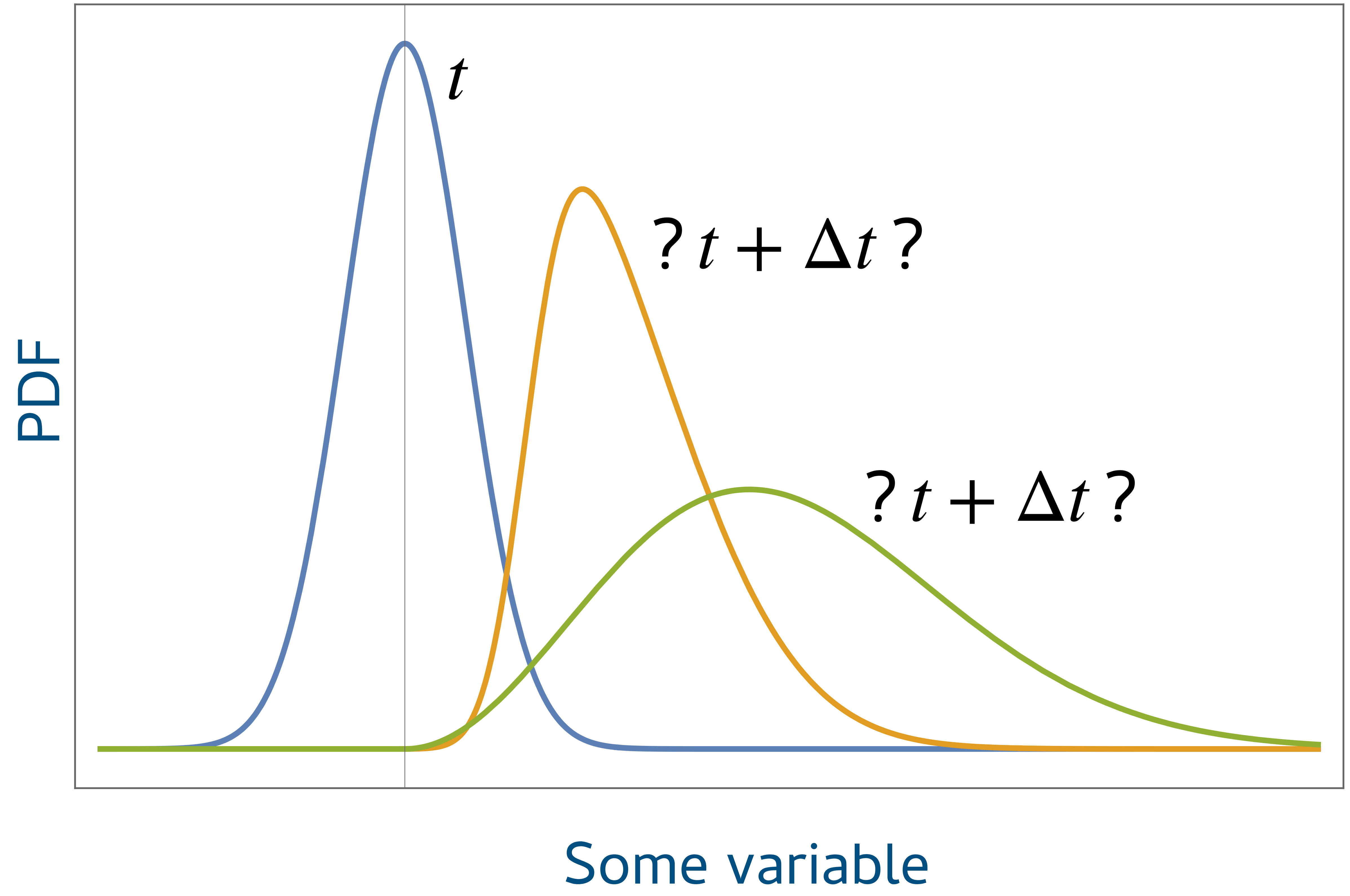
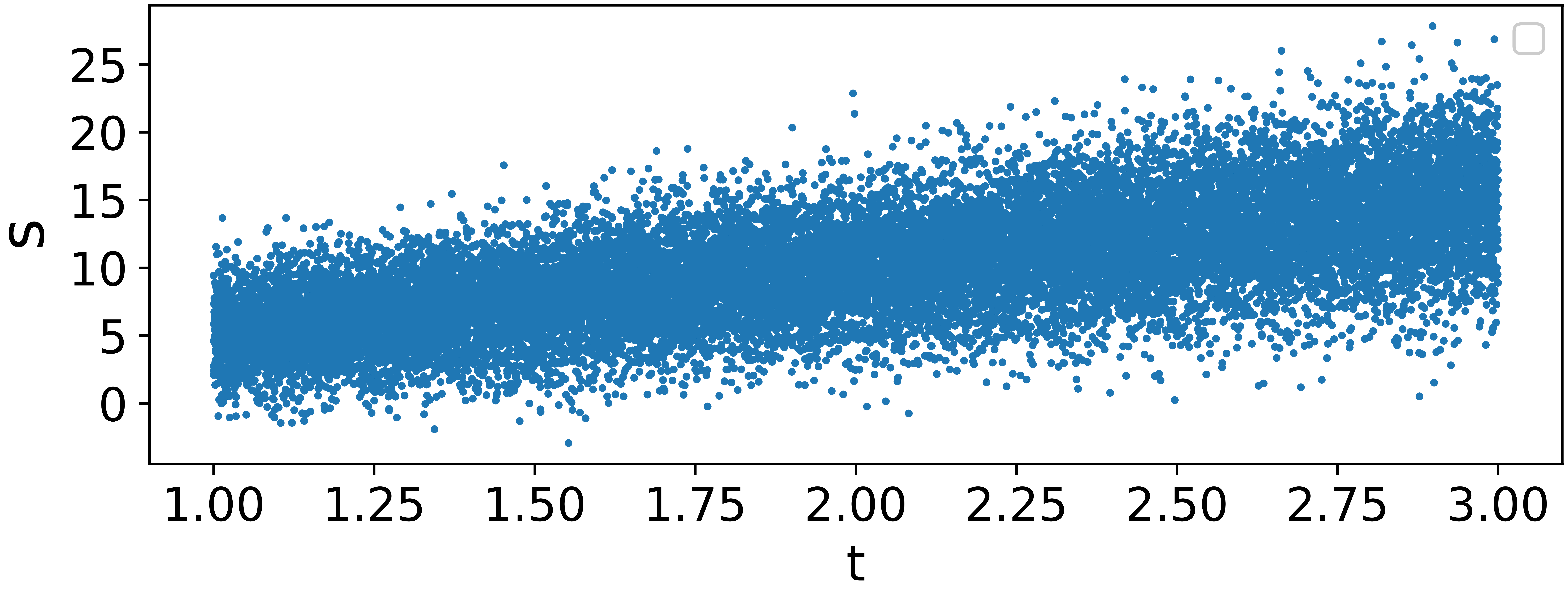


Changes in Probability distributions



Exploring changes in quantiles and moments



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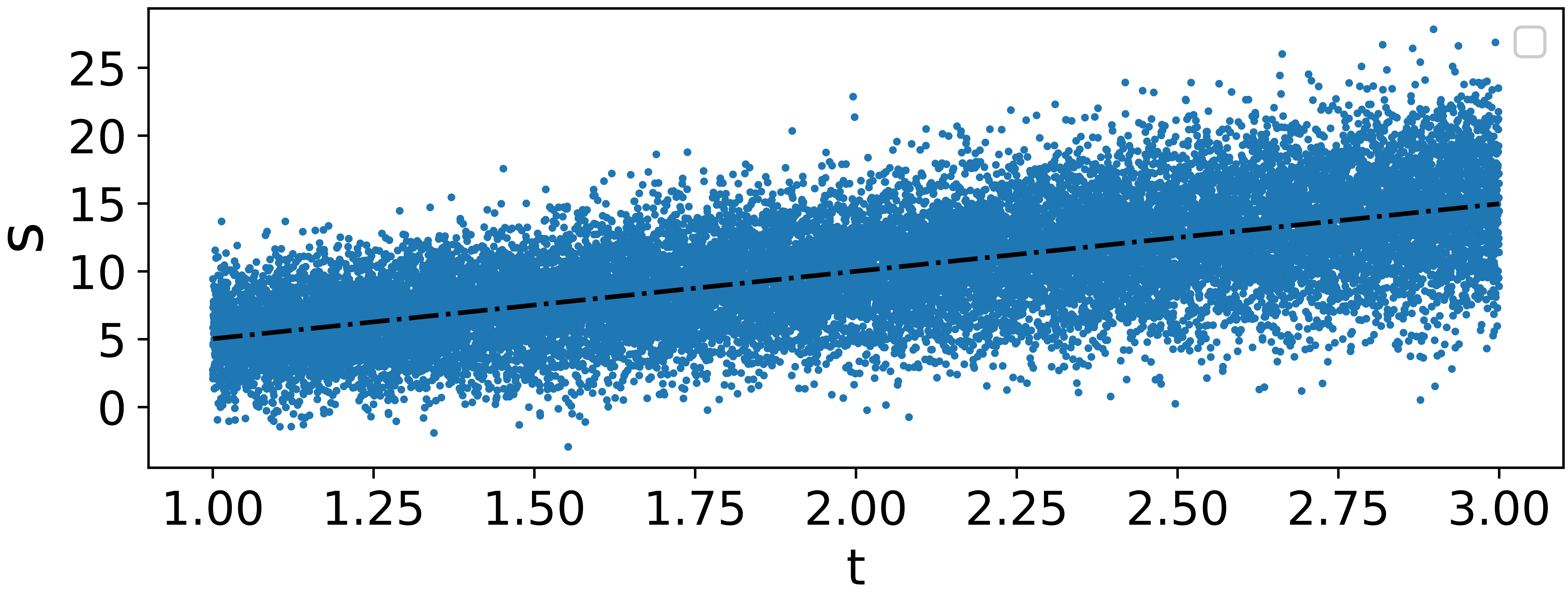

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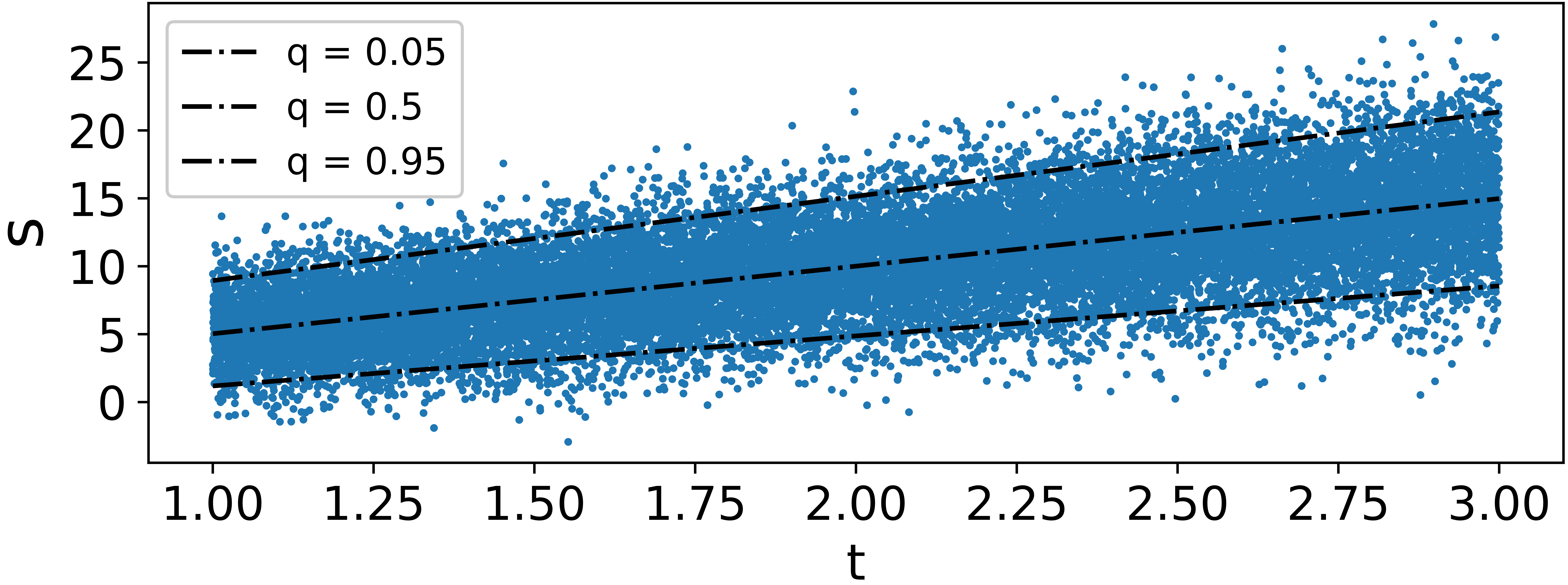
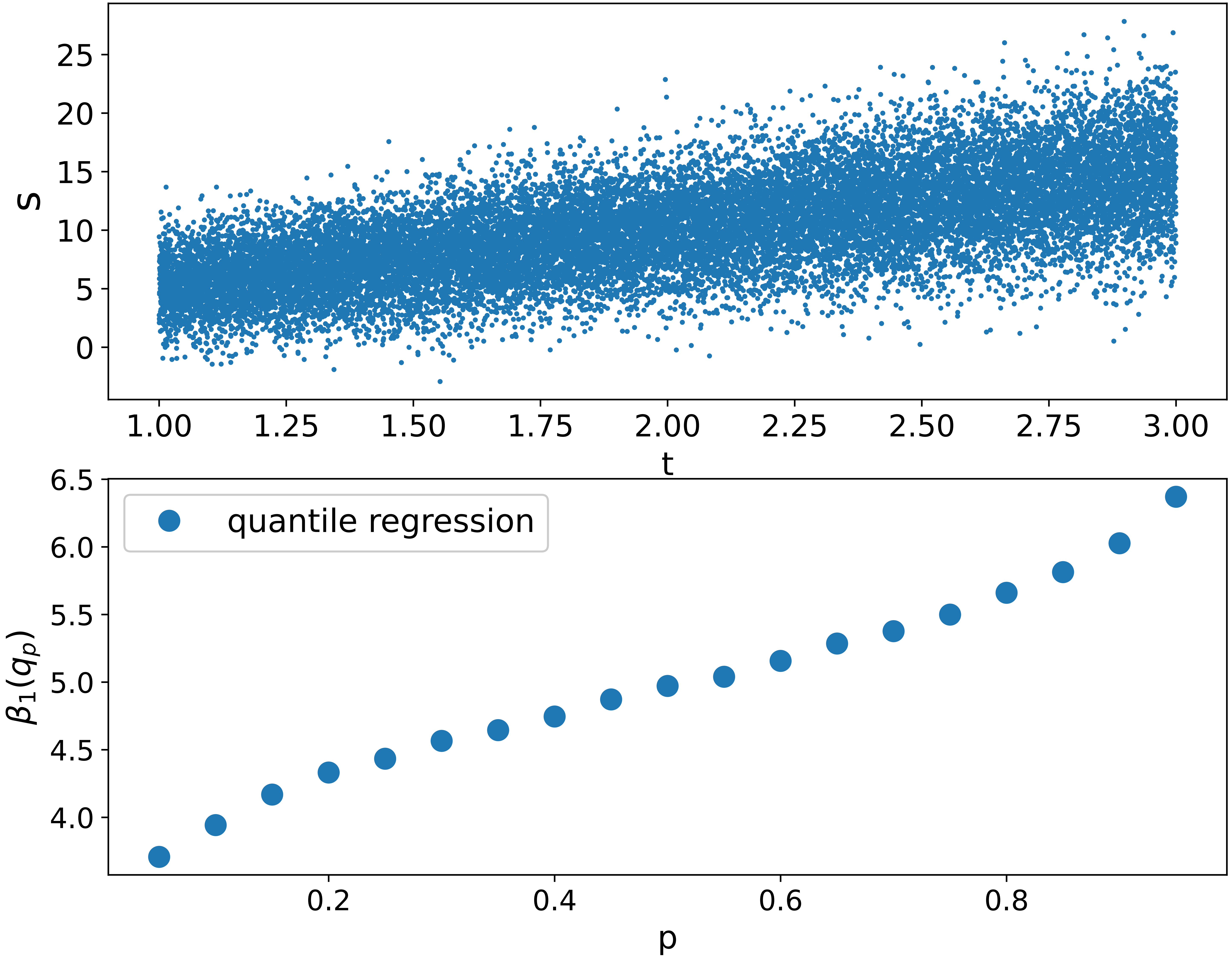


Quantile regression $\underset{\beta_0(q_p),\beta_1(q_p)\in\mathbb{R}}{\arg\min} \sum_{i=1}^{n} \rho_p(s_i - \beta_0(q_p) - \beta_1(q_p)\, t_i)$

with $\rho_p(u) = p \max(u,0) + (1-p) \max(-u,0)$; with $p \in (0,1)$

Exploring changes in quantiles and moments



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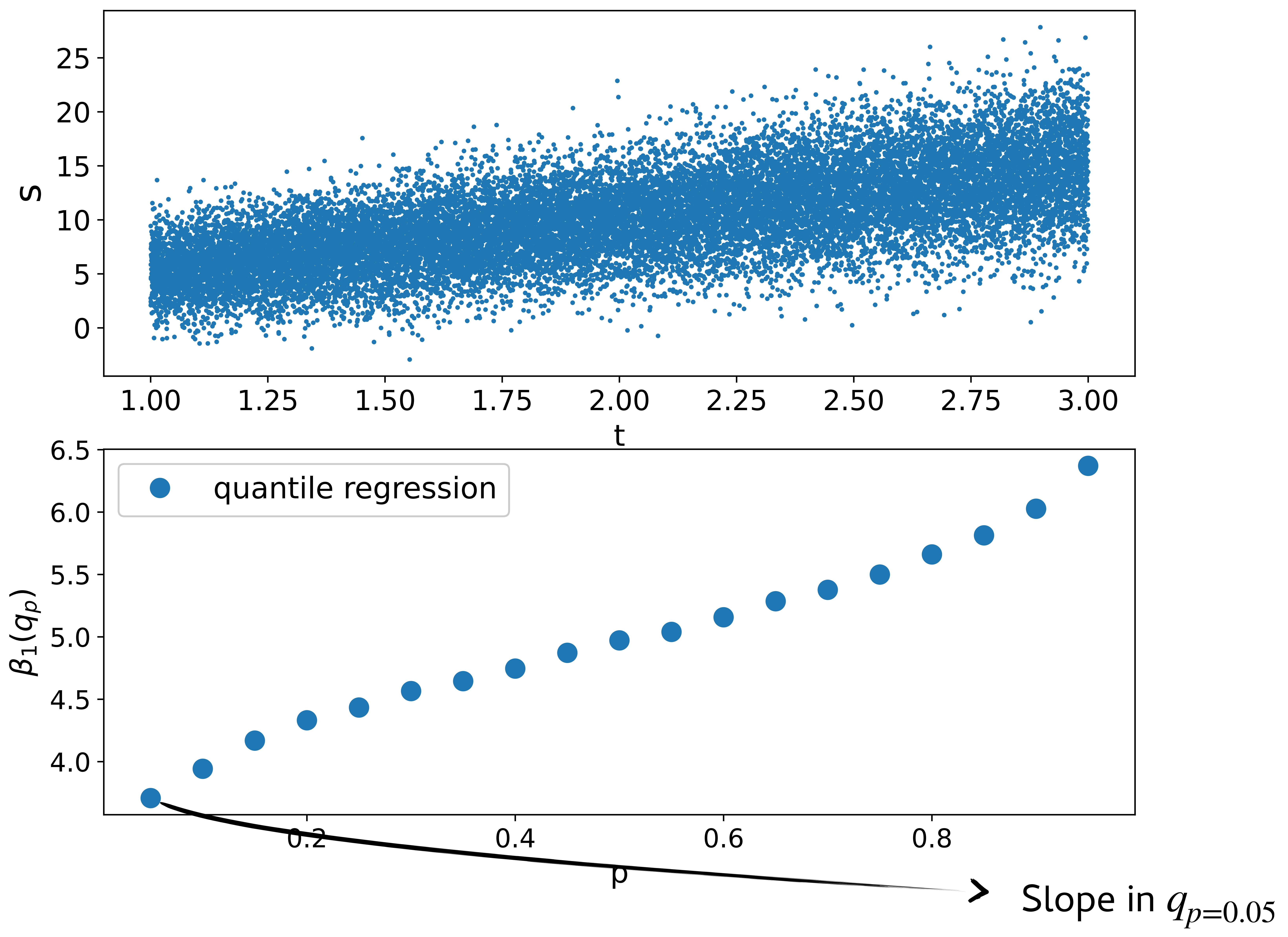

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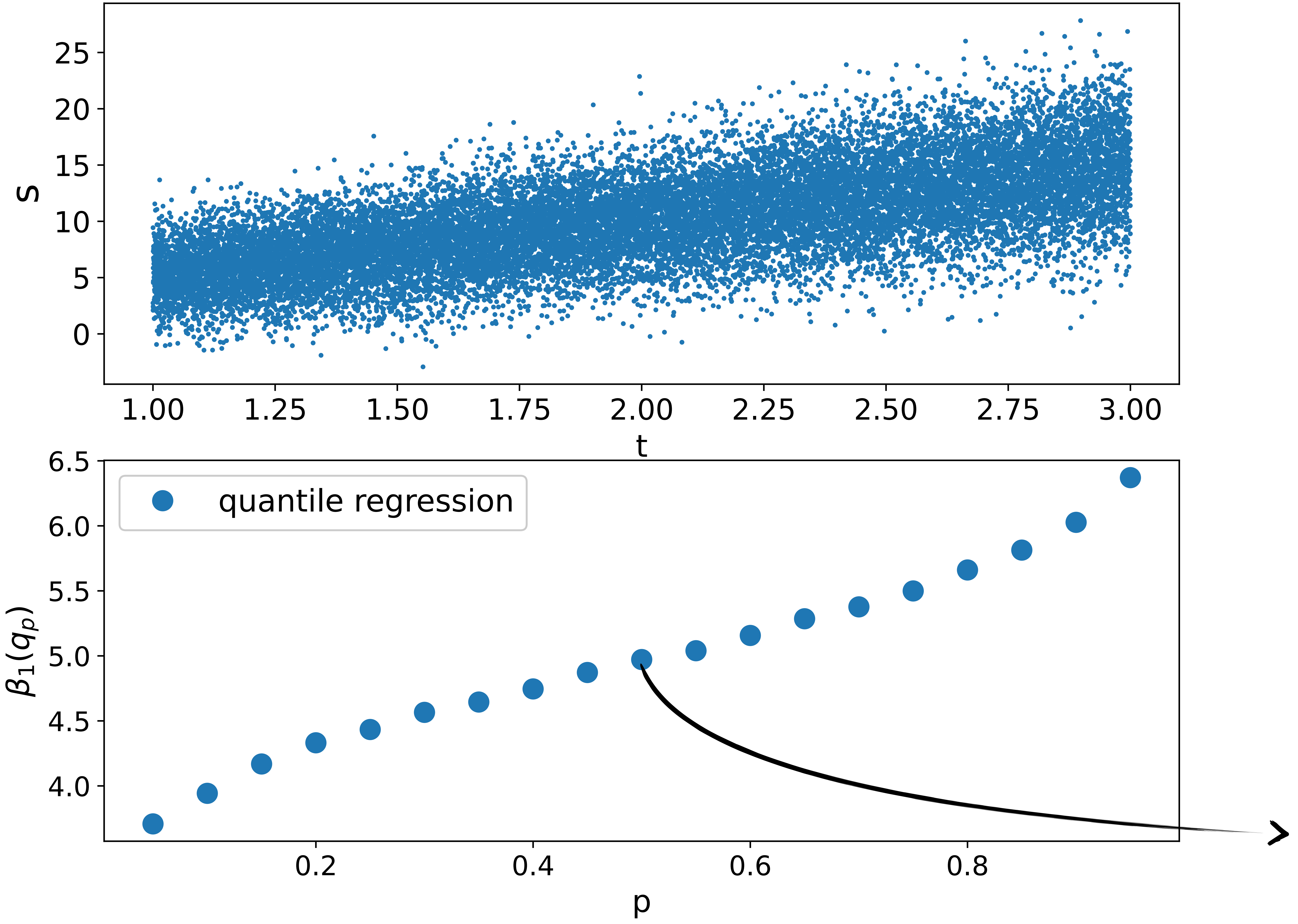
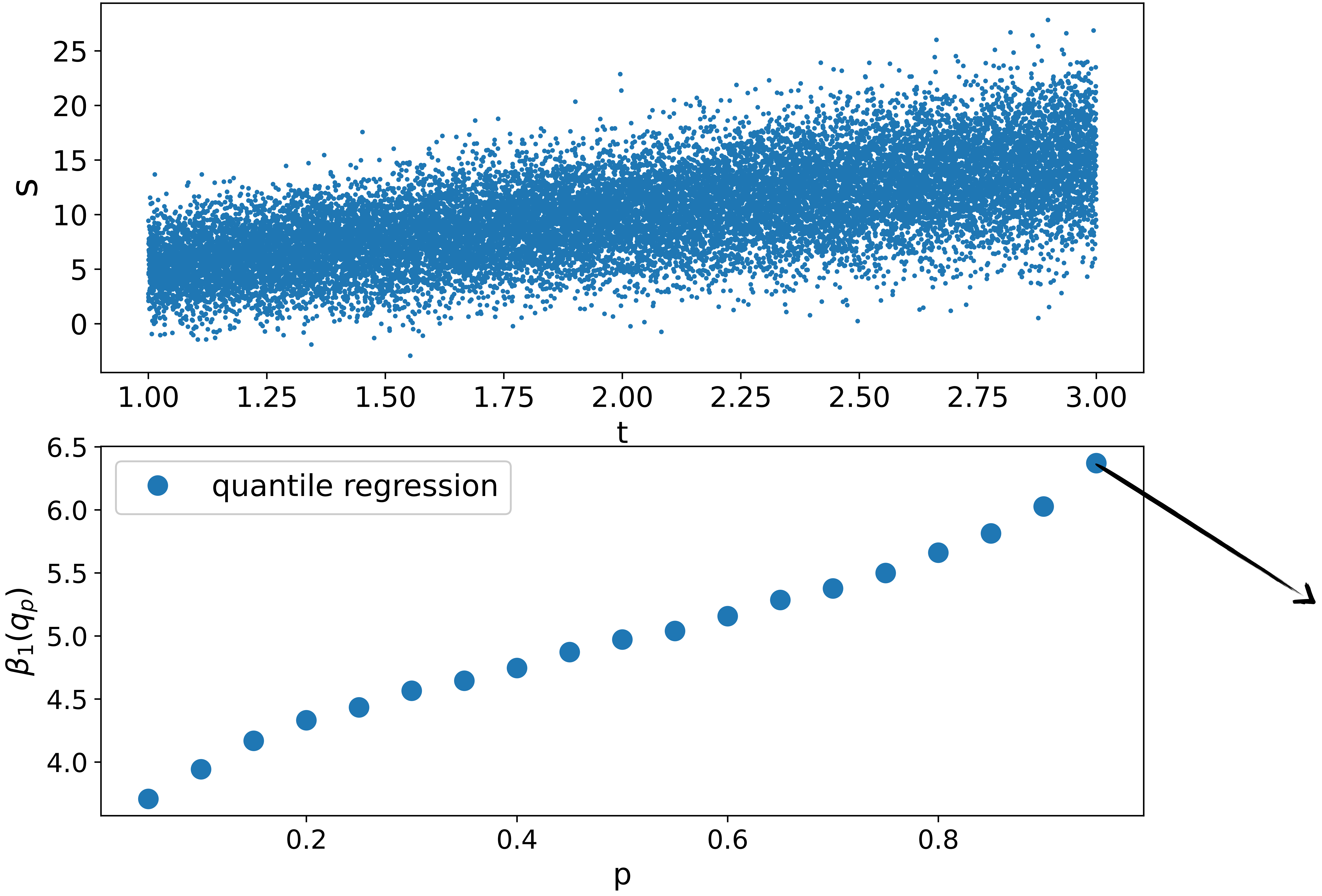


Slope in the median: $q_{p=0.5}$

Exploring changes in quantiles and moments



Slope in $q_{p=0.95}$

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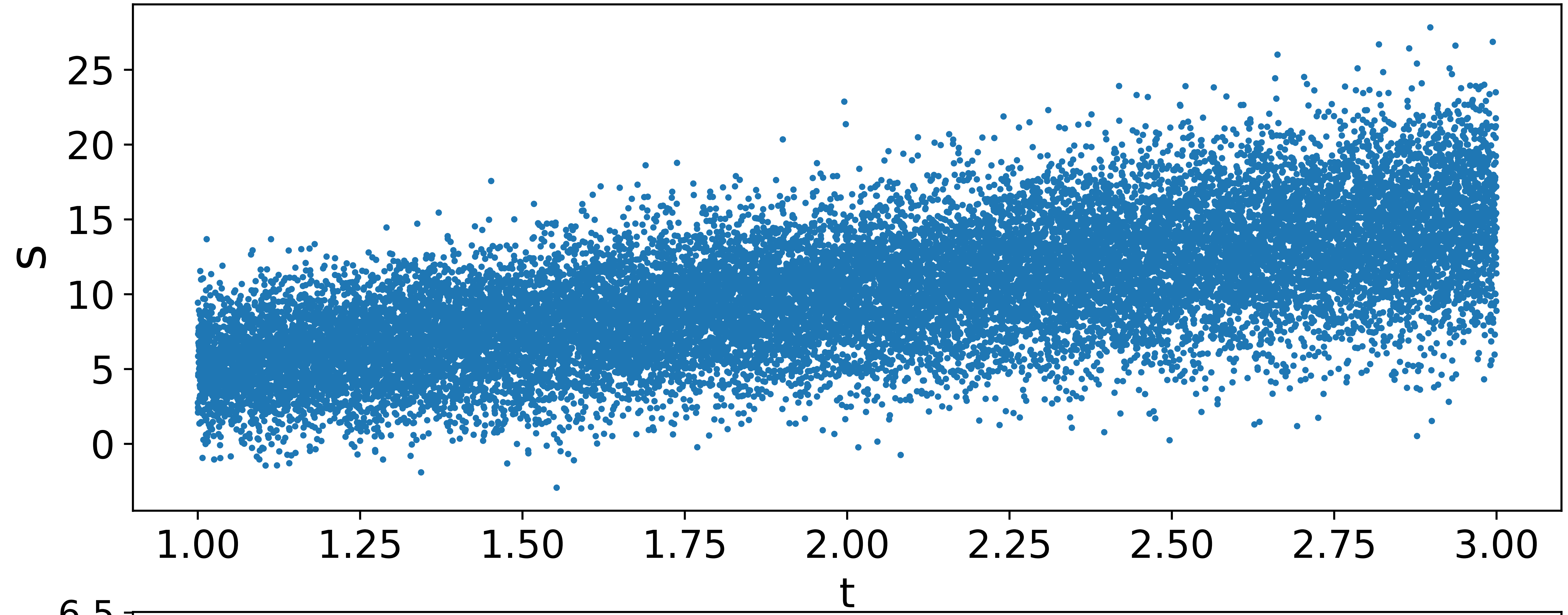
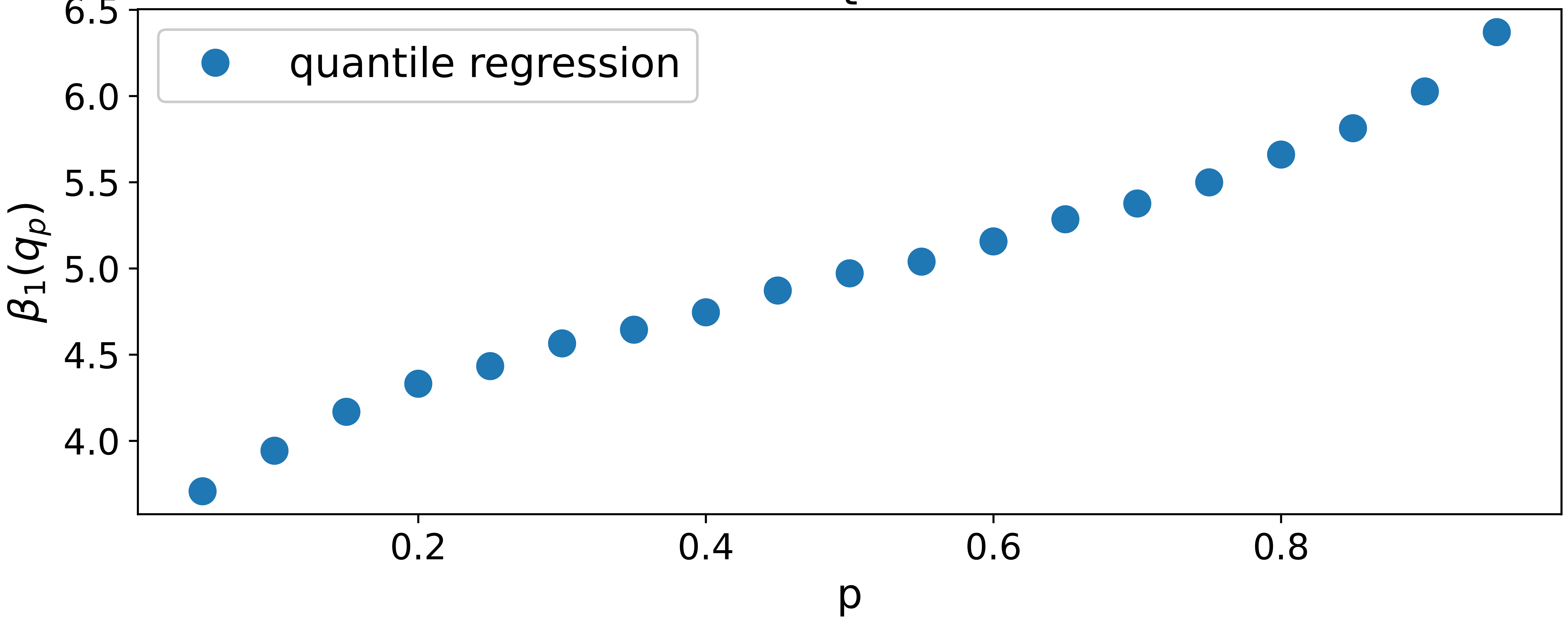
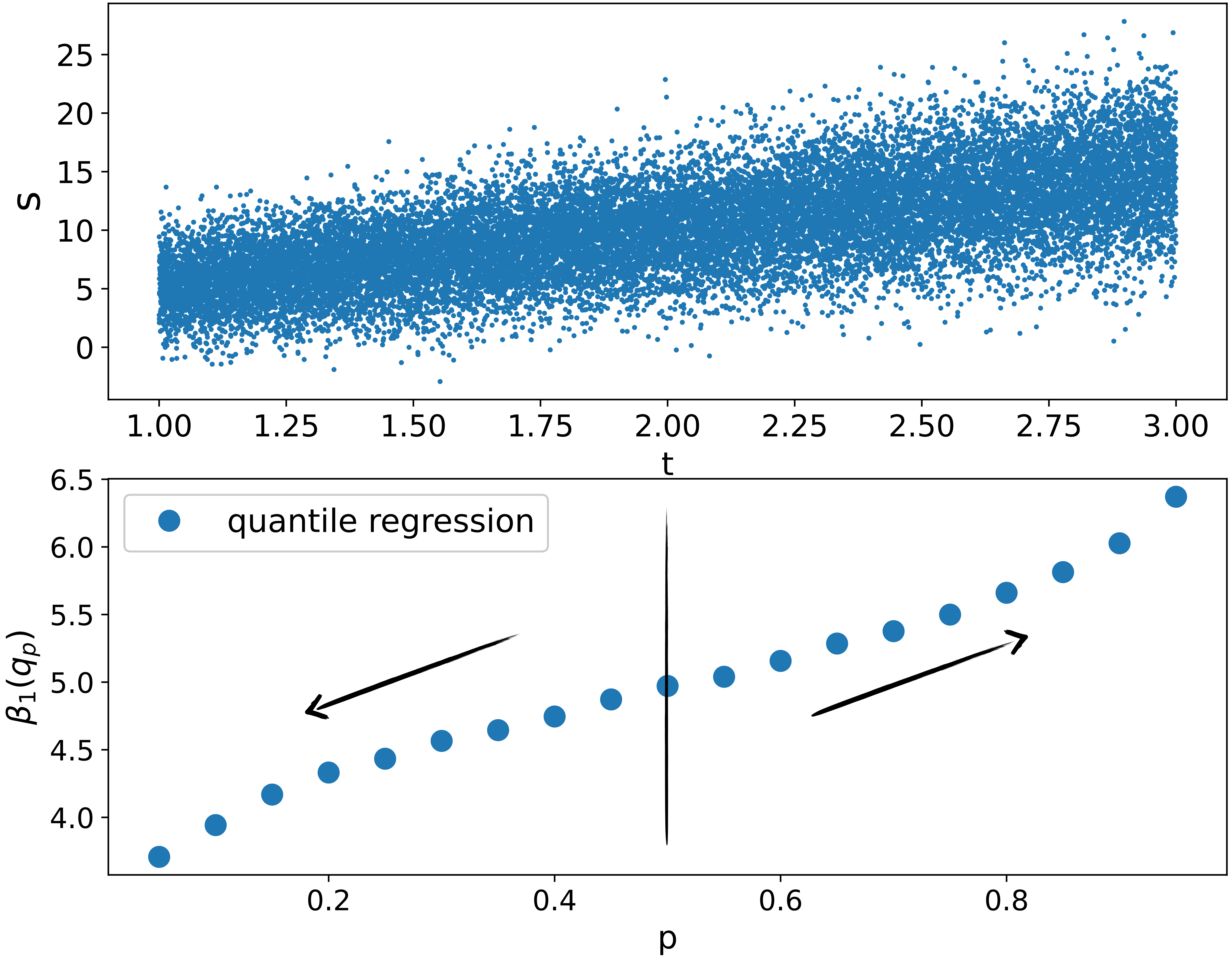




- All quantiles slopes $\beta_1(q_p) > 0$: the **mean** of the distribution is increasing

Exploring changes in quantiles and moments



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- The **variance** seems to be changing too:
  - Changes are symmetric over the median $q_{0.5}$
  - Larger quantiles are changing faster
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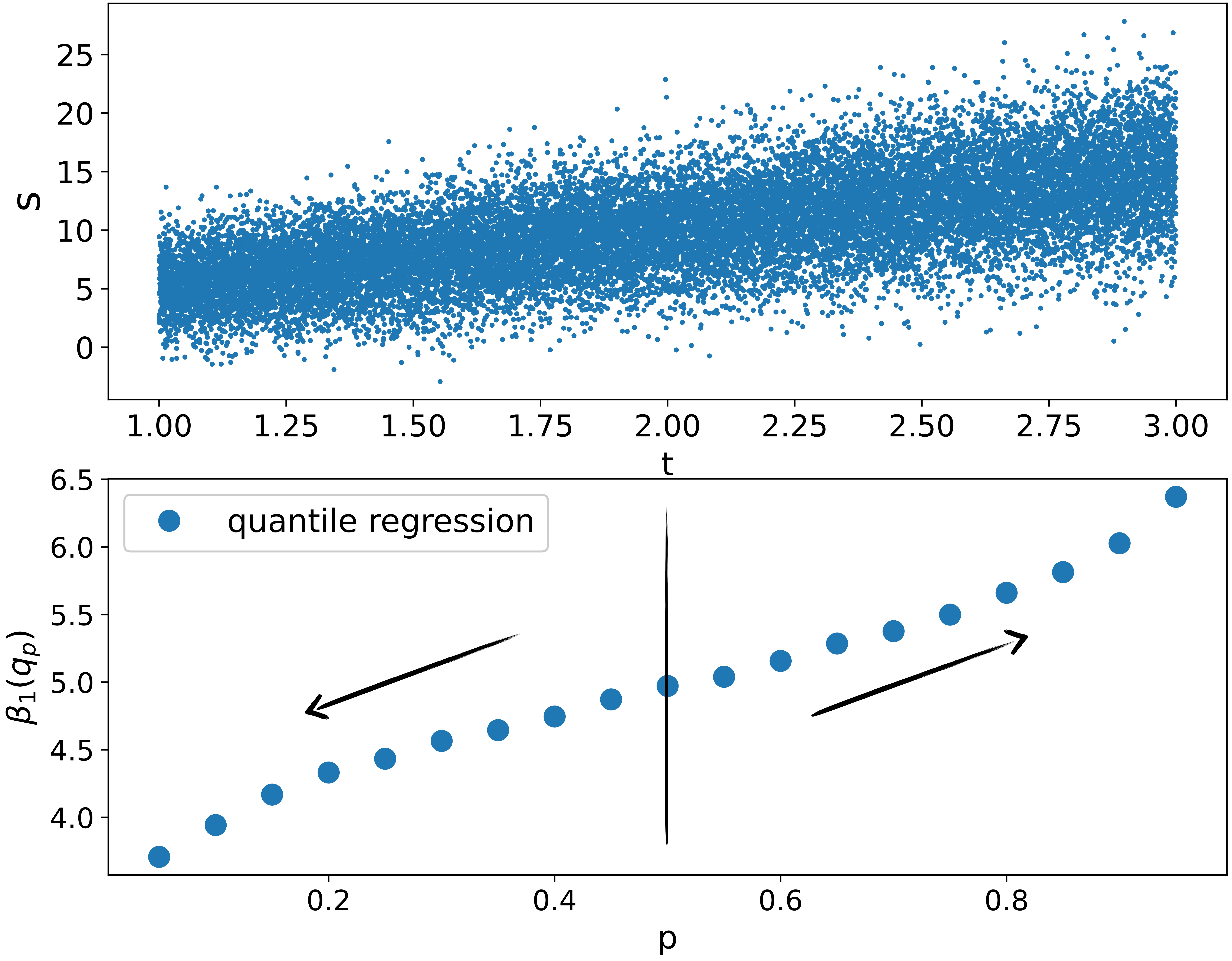

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- How to deal with N $(N >> 1)$ time series?
- Statistical significance?

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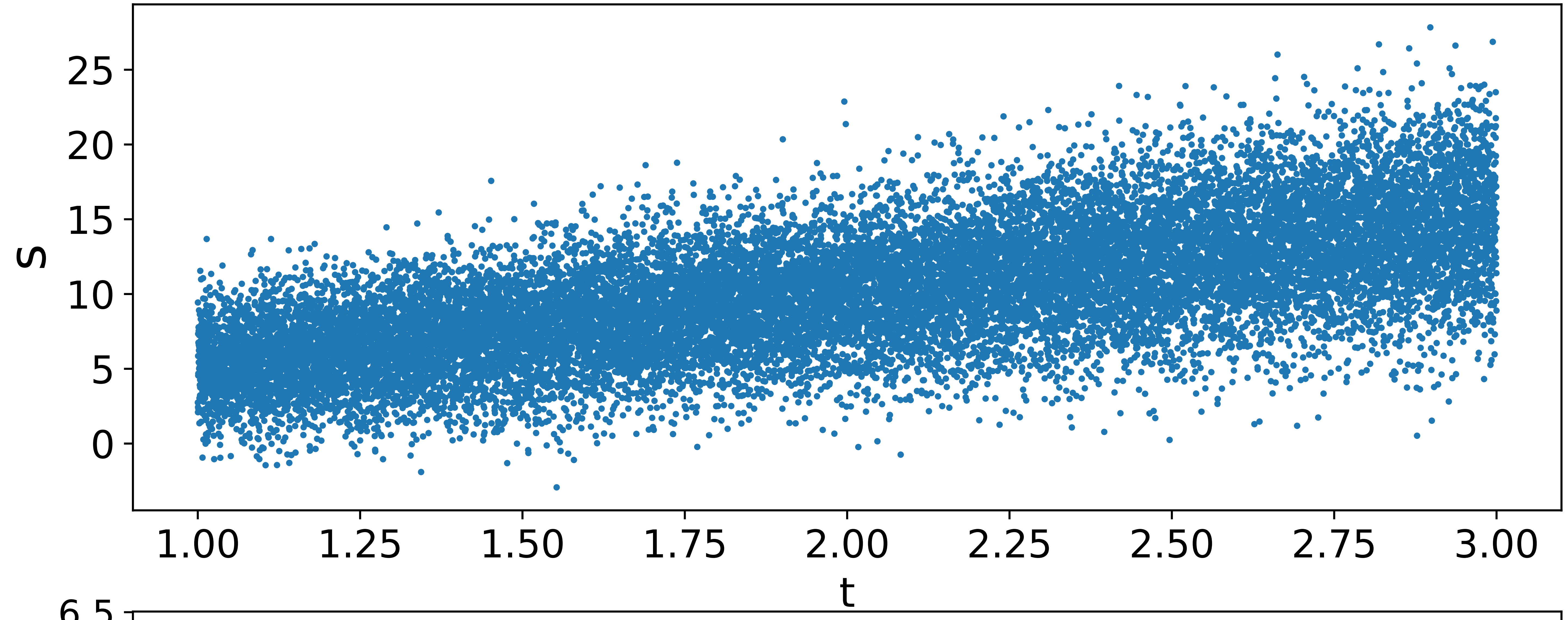

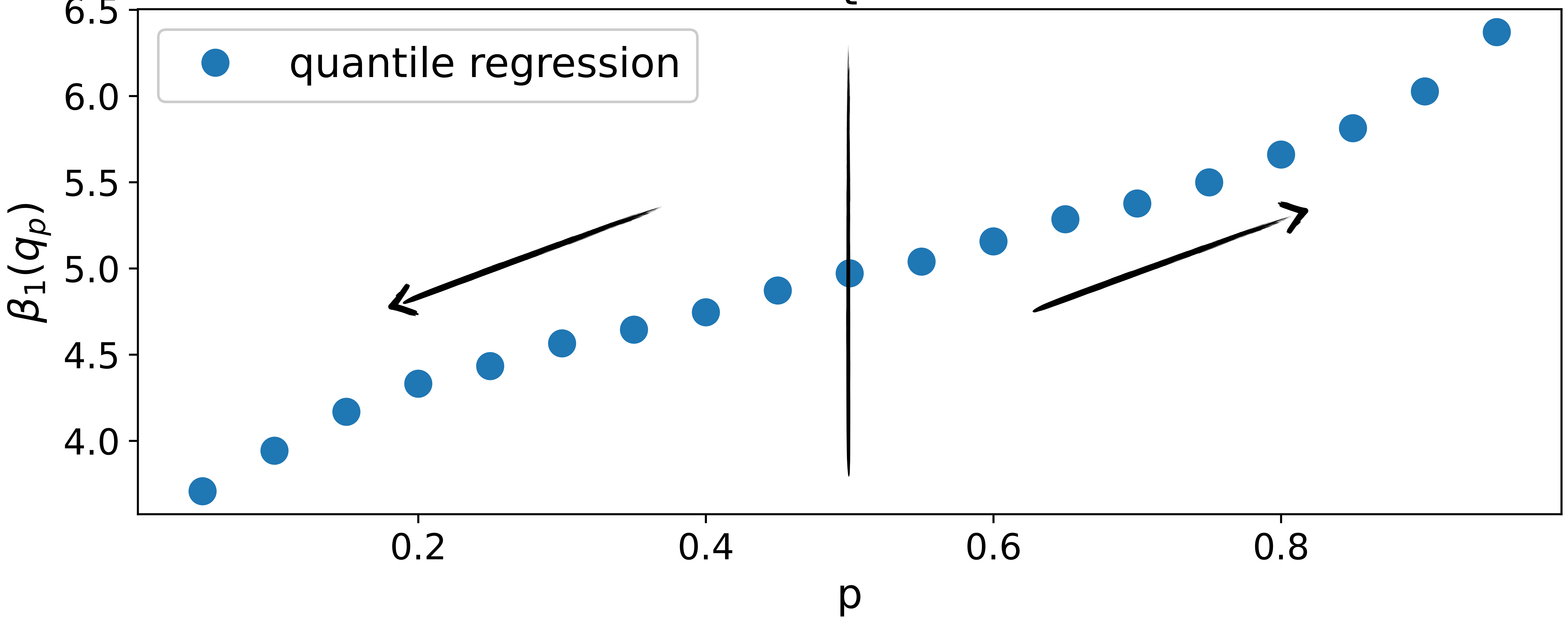

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Strategy

Construct a framework to link changes in quantiles and moments of a distribution.

McKinnon, K et al. The changing shape of NH summer temperature distributions, JGR (2016)

Exploring changes in quantiles and moments

(i.e., Median: $q_{p=0.5}$)

$$q_p(t) = q_p(m_1(t), m_2(t), m_3(t), m_4(t))$$

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Focus on Linear Changes

$$\frac{dq_p}{dt} = \frac{\partial q_p}{\partial m_1}\frac{dm_1}{dt} + \frac{\partial q_p}{\partial m_2}\frac{dm_2}{dt} + \frac{\partial q_p}{\partial m_3}\frac{dm_3}{dt} + \frac{\partial q_p}{\partial m_4}\frac{dm_4}{dt} = \sum_{i=1}^{4}\frac{\partial q_p}{\partial m_i}\frac{dm_i}{dt}$$

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Assuming relative small deviation from Gaussianity

$$\beta_1(q_p) \sim \left.\frac{dq_p}{dt}\right|_{m_*} = \sum_{i=1}^{4}\frac{dm_i}{dt}\left.\frac{\partial q_p}{\partial m_i}\right|_{m_*} = \sum_{i=1}^{4}\frac{dm_i}{dt}b_i(p)$$

Cornish,E.A. & Fisher,R.A.; Revue De L'institut International De Statistique (1937)

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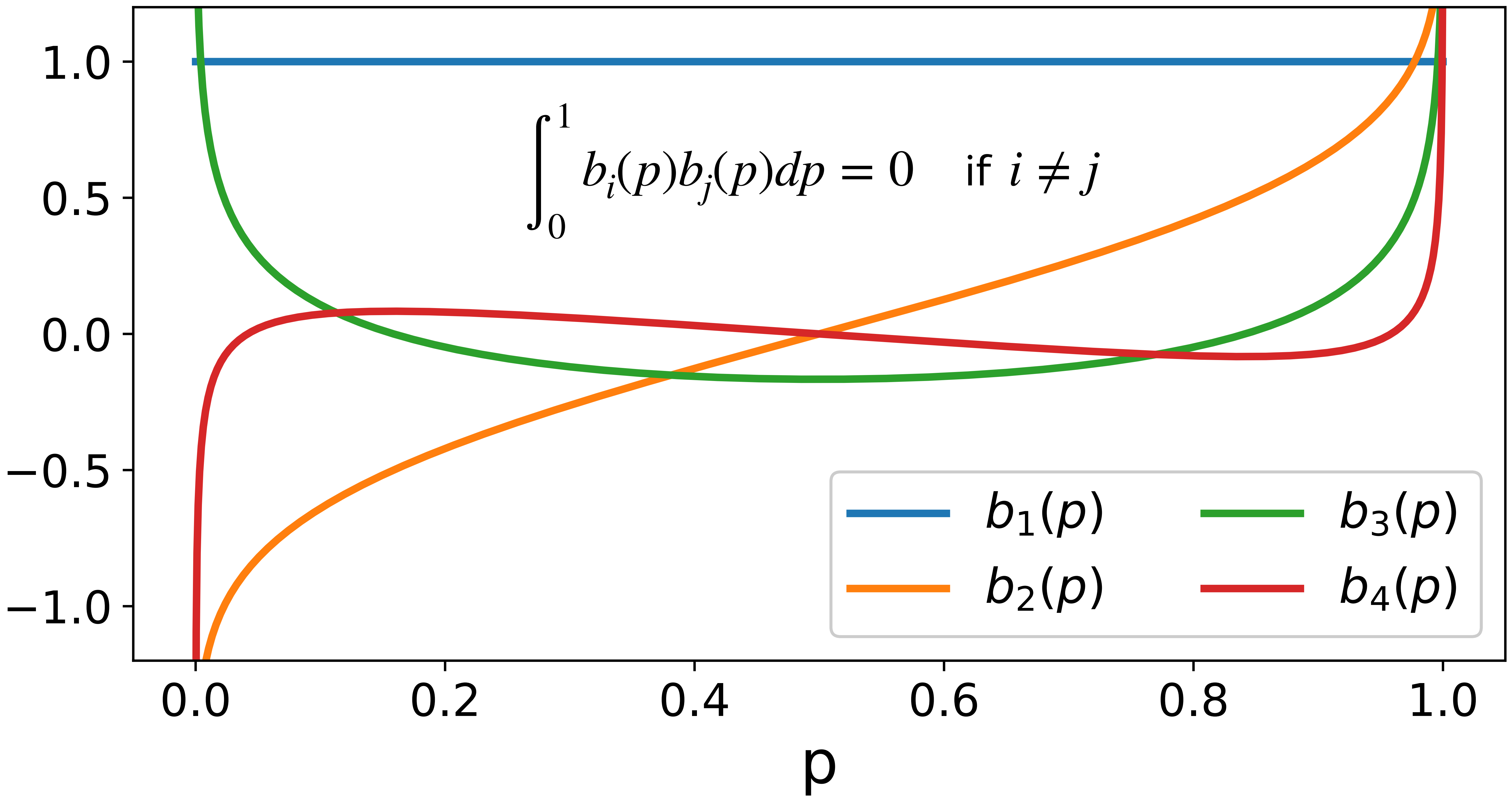
Computed by quantile regression



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Falasca et al. Exploring the non-stationarity of Coastal sea level probability distributions arXiv:2211.04608

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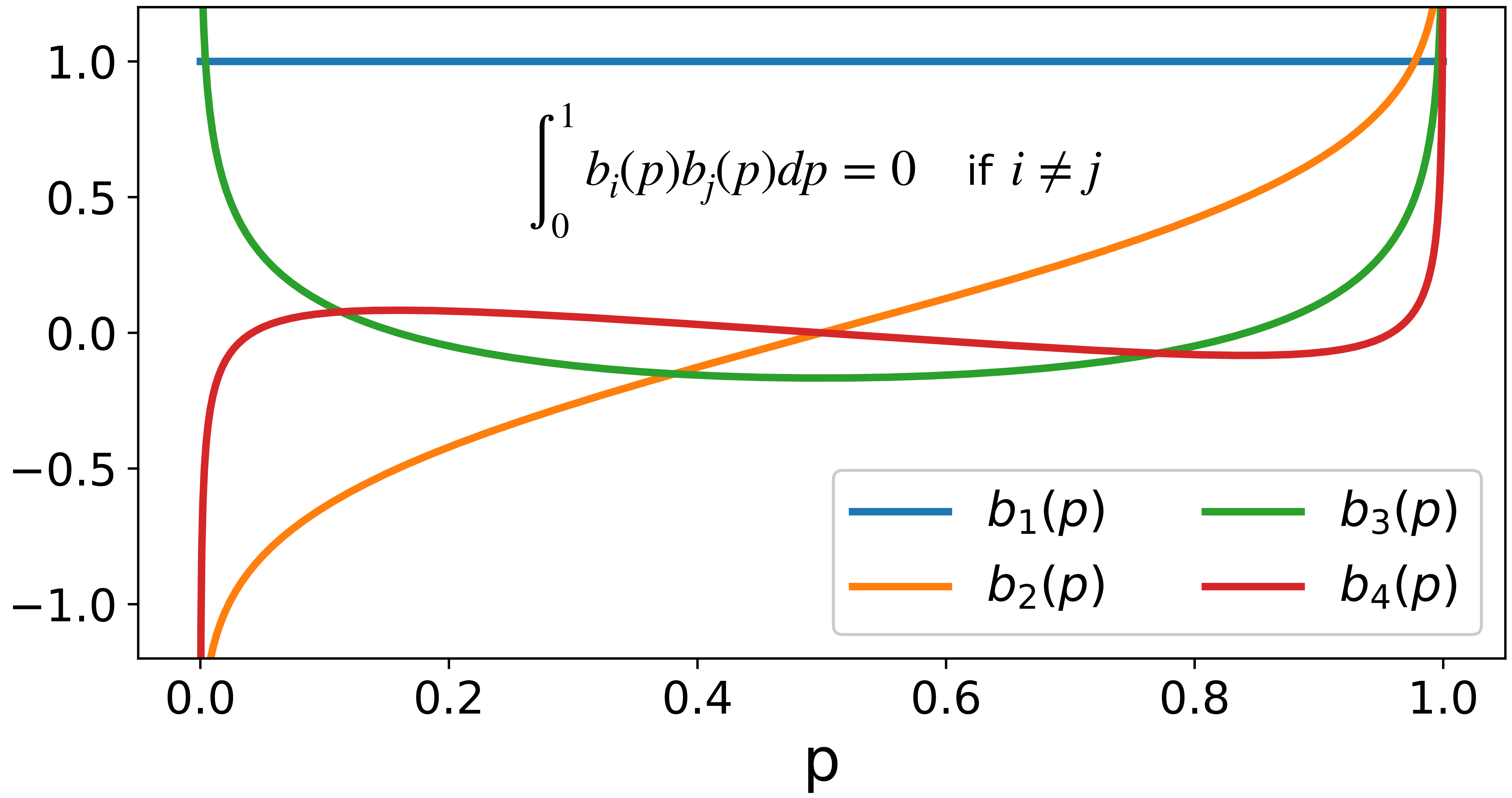
Polynomials $b_i(p)$ quantify how quantiles of a distribution change when shifting its moments one at a time



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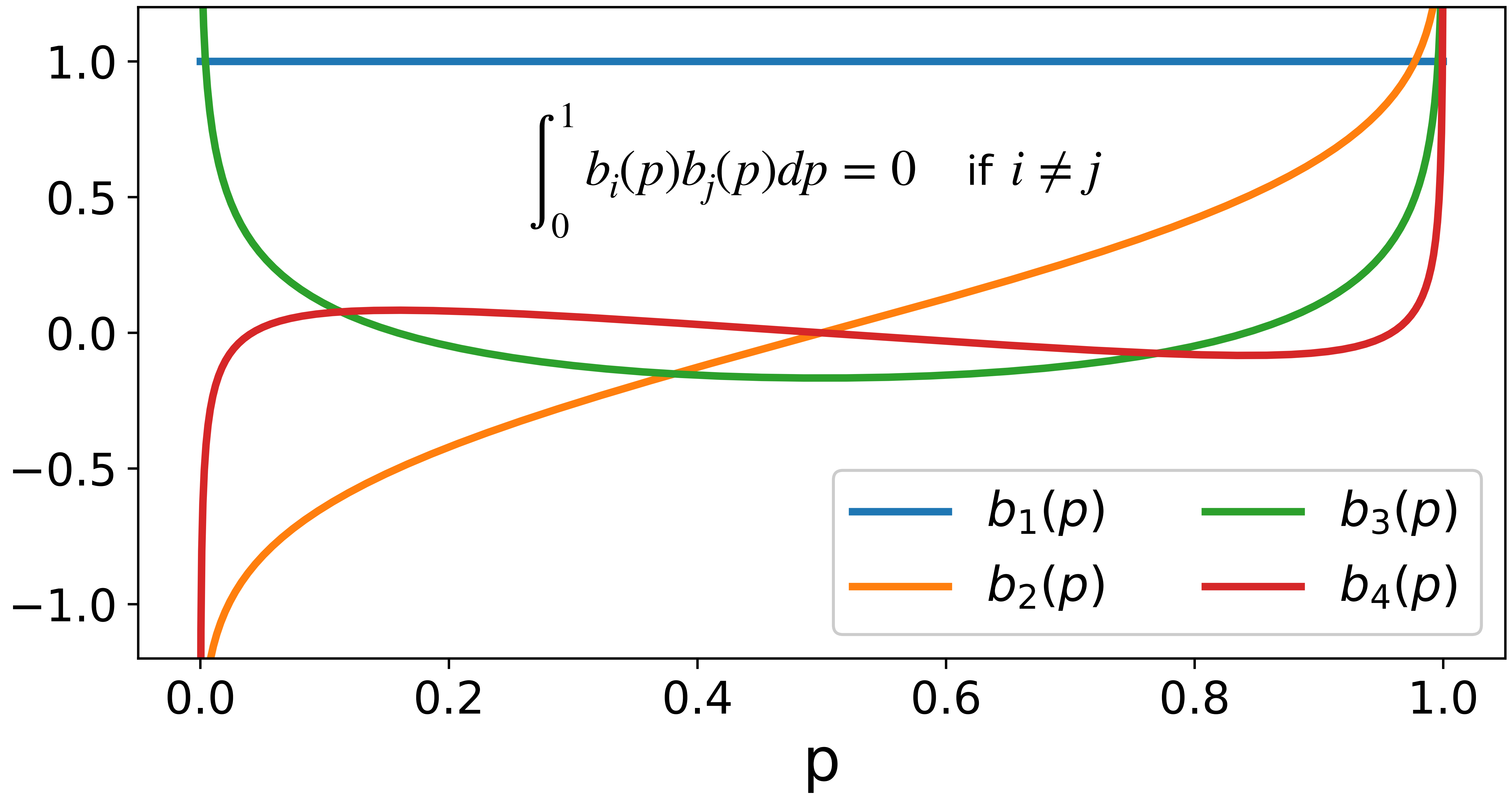

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Measuring slopes $\beta_1(q_p)$ of quantiles q_p for $p \in [0.05, 0.95]$ every $\delta p = 0.05$)

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Application to Coastal Sea Level Rise

Tide gauges
Daily sea level
Period: 1970–2017
O: significant
X: not significant
(a) Mean
(b) Variance
(c) Skewness
(d) Kurtosis
45°N
0°
45°S
180°
90°W
0°
mm yr⁻¹
mm² yr⁻¹
yr⁻¹
4
2
0
−2
−4


Tide gauges
Daily sea level
T > 80yrs
O: significant
X: not significant
(a) Mean
(b) Variance
(c) Skewness
(d) Kurtosis
45°N
0°
45°S
180°
90°W
0°
mm yr^{-1}
mm^2 yr^{-1}
yr^{-1}
4
2
0
−2
−4

Sea level decomposition

$$\Delta\eta = \eta_{dyn} + \eta_{ib} = \frac{\Delta P_b}{\rho_0\, g} - \frac{\Delta P_a}{\rho_0\, g} - \frac{1}{\rho_0}\int_{-H}^{\eta} \Delta\rho \, \mathrm{d}z$$

Griffies S.M. and Greatbatch R.J. *Ocean Modeling (2012)*

Where:

- $\eta_{dyn} = \frac{\Delta P_b}{\rho_0\, g} - \frac{1}{\rho_0}\int_{-H}^{\eta} \Delta\rho \, \mathrm{d}z$
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Changes in water column mass:

- Convergence of mass via ocean currents
- Water crossing the ocean free surface

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Local steric effects:

- Changes in sea level driven by changes in density

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Inverse Barometer:

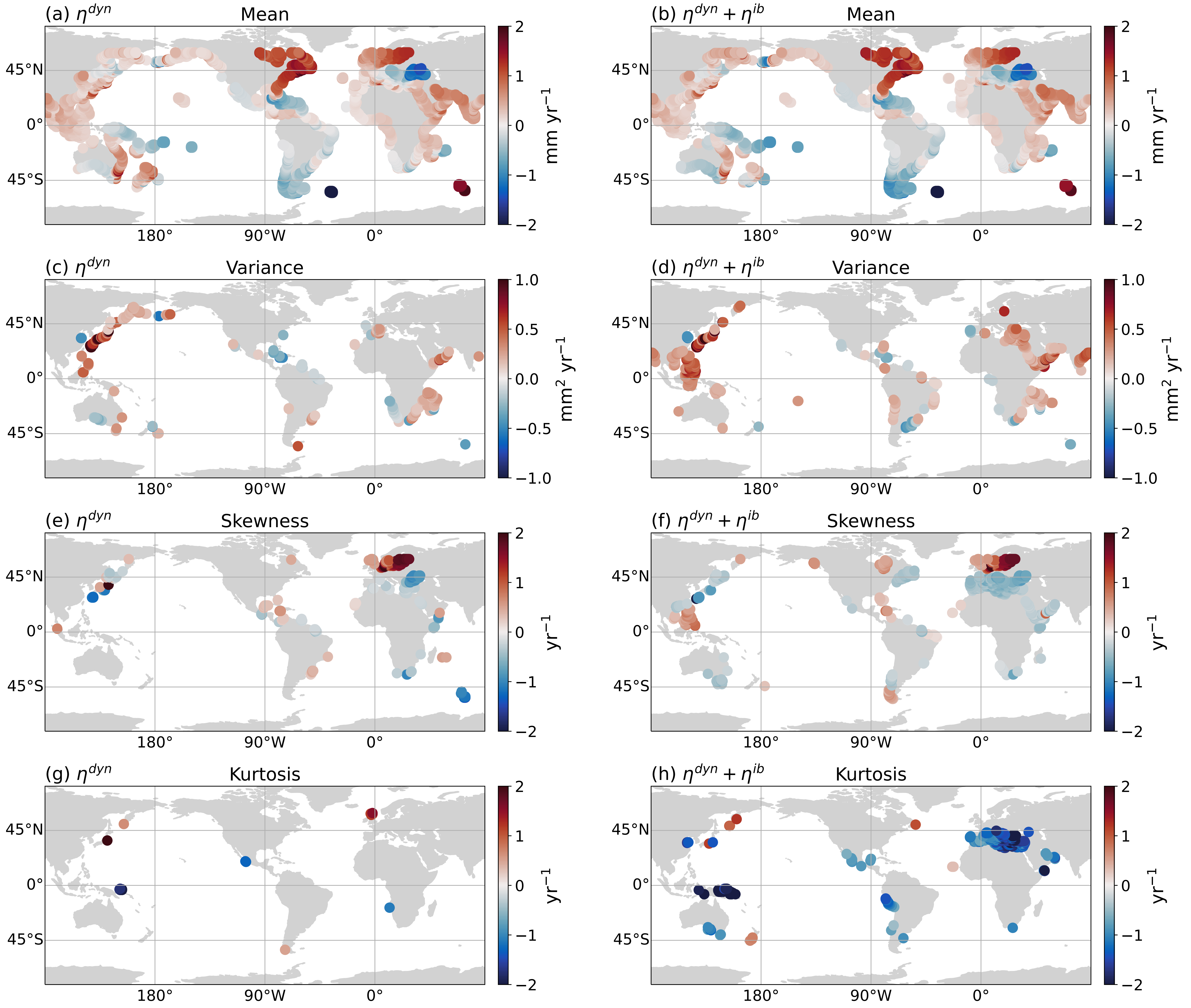
- Changes in sea level driven by local changes in sea level pressure

GFDL-CM4: Historical
Daily sea level
Historical run: 1970–2014
O: significant
(a) $\eta^{dyn} + \eta^{ib}$ Mean
mm yr^{-1}
(b) $\eta^{dyn} + \eta^{ib}$ Variance
mm^2 yr^{-1}
(c) $\eta^{dyn} + \eta^{ib}$ Skewness
yr^{-1}
(d) $\eta^{dyn} + \eta^{ib}$ Kurtosis
yr^{-1}
45°N
0°
45°S
180°
90°W
0°

GFDL-CM4: 1pctCO_2 Experiment

Daily sea level

O: significant

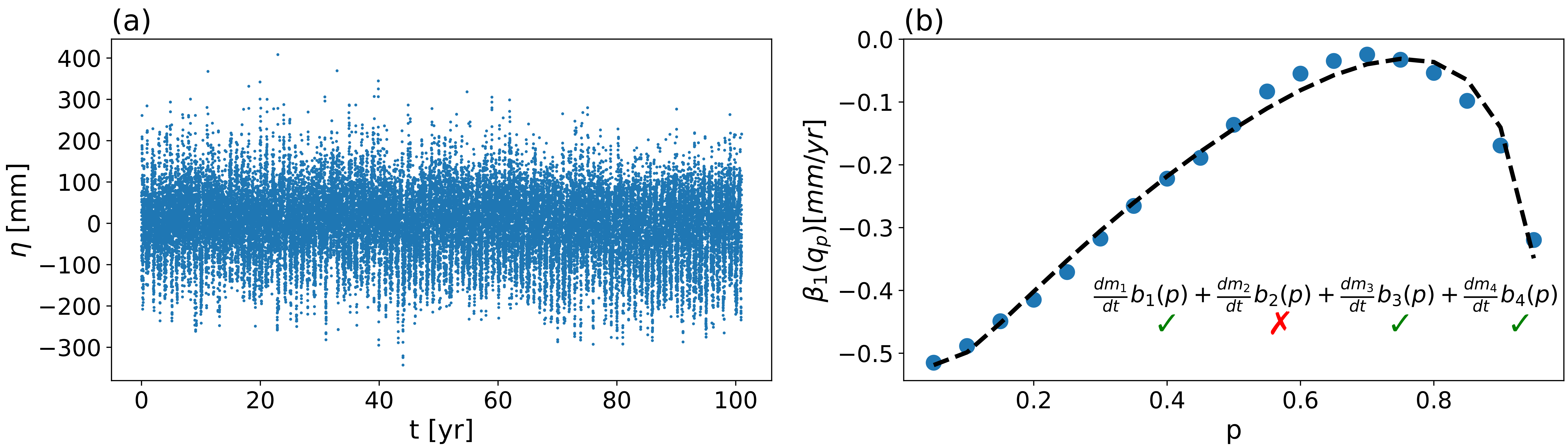


1pctCO_2 Experiment

- Emergence of changes in shapes of the distribution
- Changes in variance and skewness are already present in η_{dyn}
- Changes in higher order moments are always amplified when adding η_{ib}

Daily sea level

GFDL-CM4: 1pctCO$_2$ Experiment



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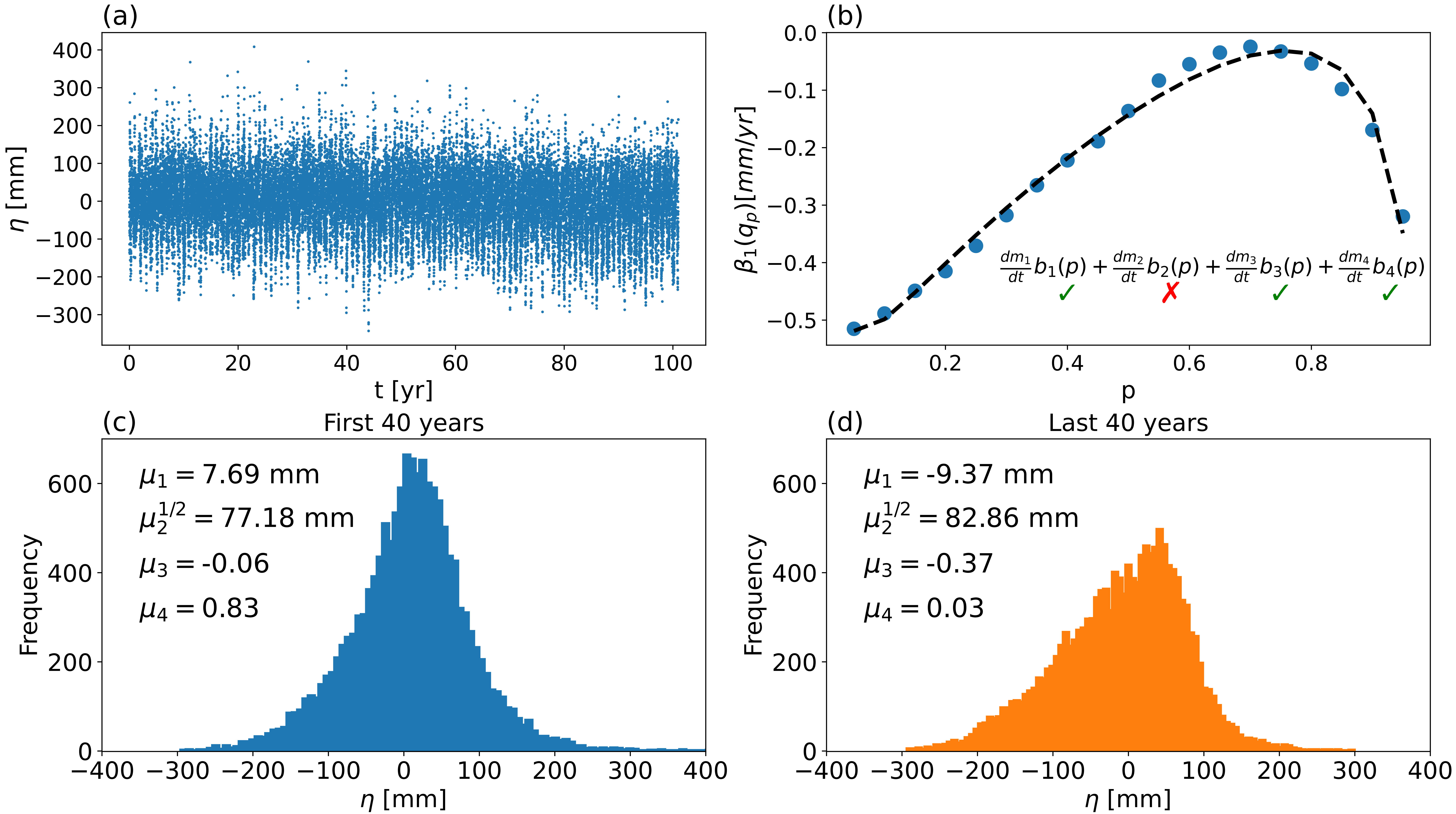


Conclusions

- Proposal of a general statistical model to study (significant) changes in both quantiles and moments of a distribution.
    - This is done through projection over suitable orthogonal polynomials: the methods captures *independent* sources of changes in distributions.

- Changes in daily coastal sea level in observations can be explained by solely by a shift in the mean of the distribution. The CM4 model agrees with observations in the historical period.

- In the 1%/yr CO_2 run we identify the emergence of changes in higher order moments.
    - Changes are already present in the dynamic sea level and get always amplified when the inverse barometer is included

- **Next steps:** adopting this methodology to study changes in (a) sea level distributions in open ocean across different models and scenarios and (b) application across different variables.

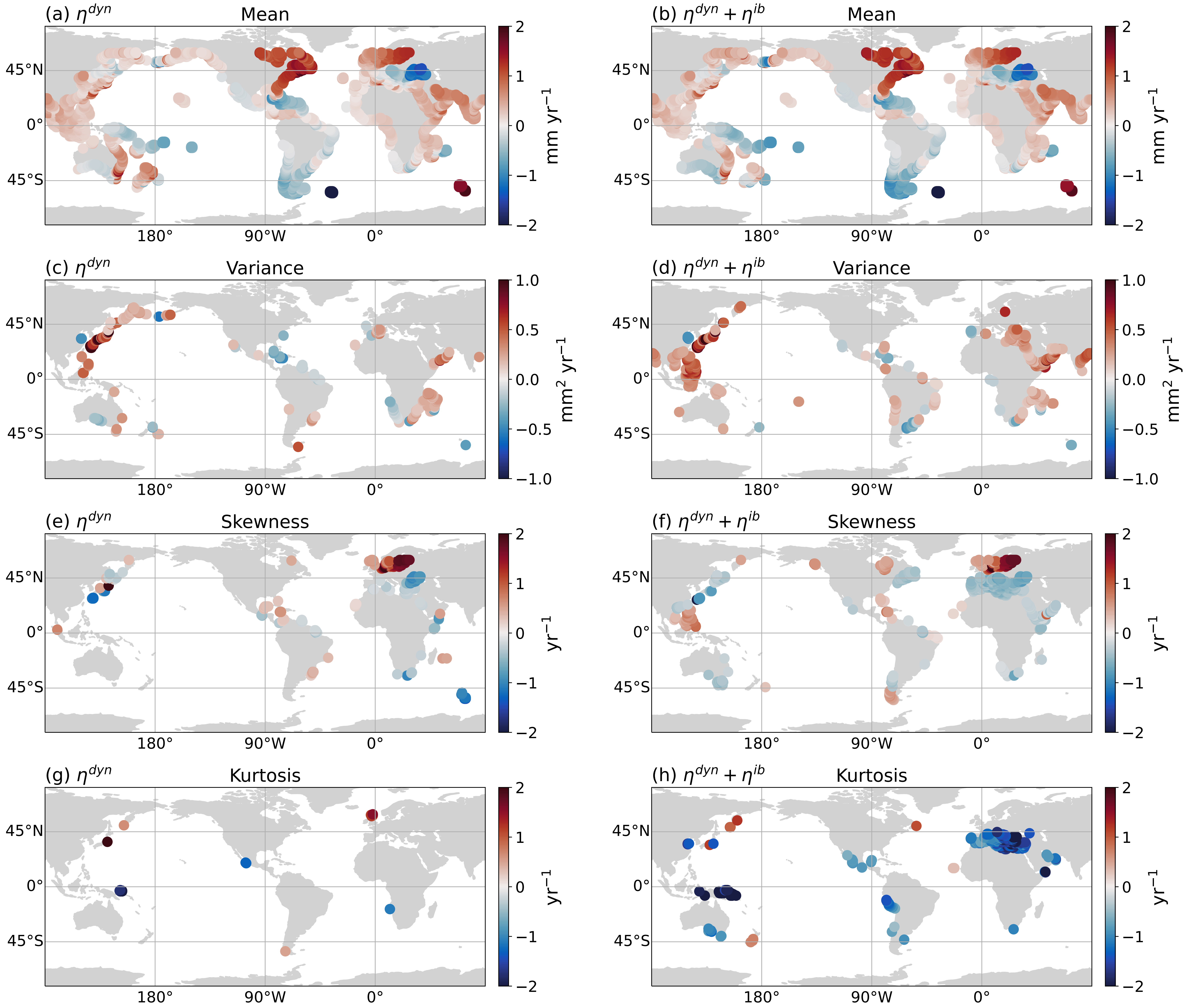
Thanks

Falasca, F. et al. Exploring the non-stationarity of coastal sea level distributions; arXiv:2211.04608

Backups

GFDL-CM4: 1pctCO$_2$ Experiment

Daily sea level



O: significant

1pctCO$_2$ Experiment

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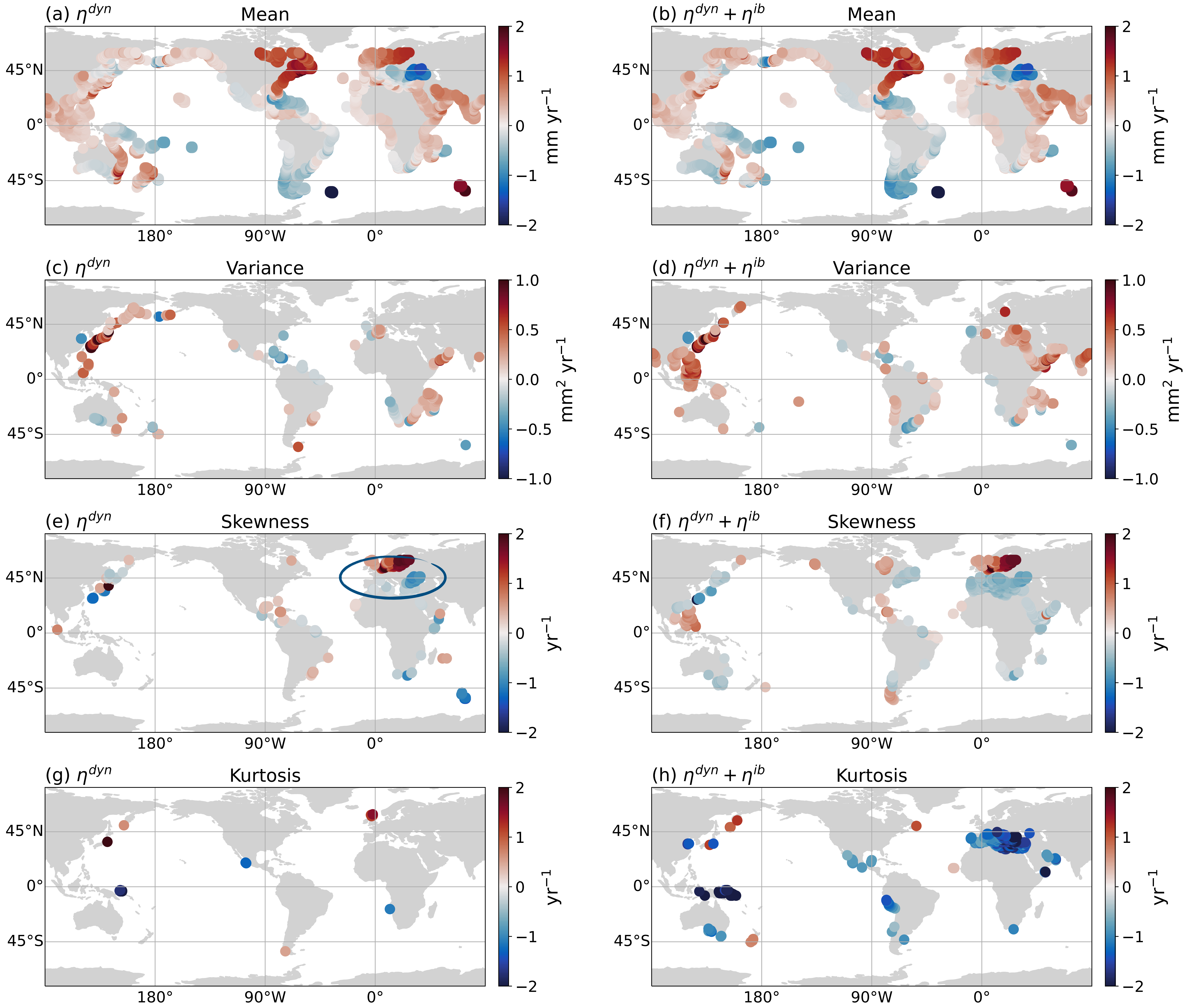
Shifts in Kurtosis only in the inverse barometer component. Possibly in agreement with Priestly and Catto (2022) who observed a decrease in Cyclone numbers in CMIP6 projections.

GFDL-CM4: 1pctCO$_2$ Experiment

Daily sea level

O: significant

1pctCO$_2$ Experiment



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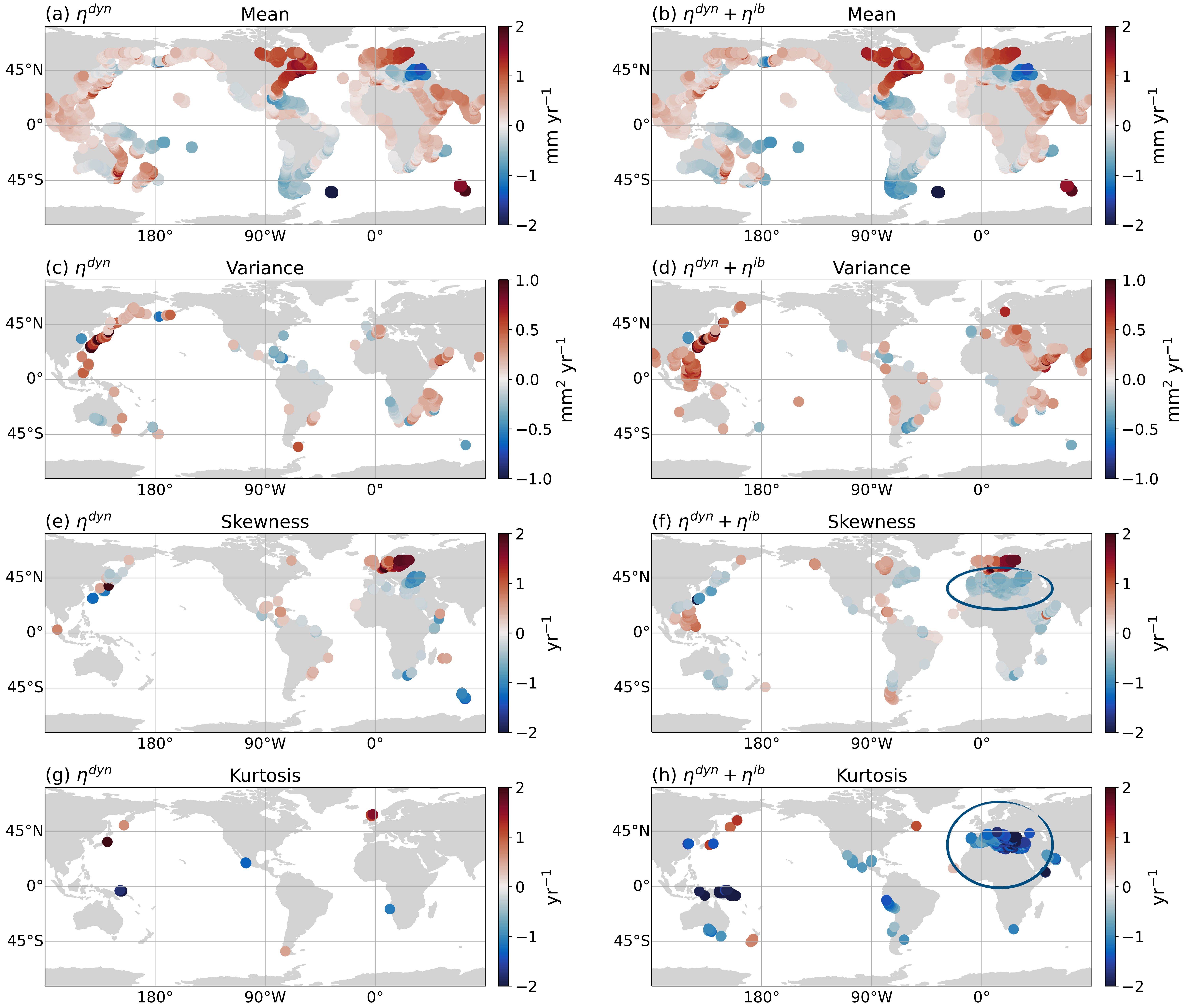
Large increase in Skewness already present in dynamic sea level only. Consistent with an increase in frequency of intense westerly winds in that region. Pinto et al. (2007)

GFDL-CM4: 1pctCO$_2$ Experiment

Daily sea level

O: significant

1pctCO$_2$ Experiment



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Shifts in Skewness and Kurtosis in the Mediterranean. Possibly pointing to a decrease in Medicanes as suggested in Gonzáles-Aléman et al. (2019)

Exploring changes in quantiles and moments

(i.e., Median: $q_{p=0.5}$)

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We focus on linear changes:

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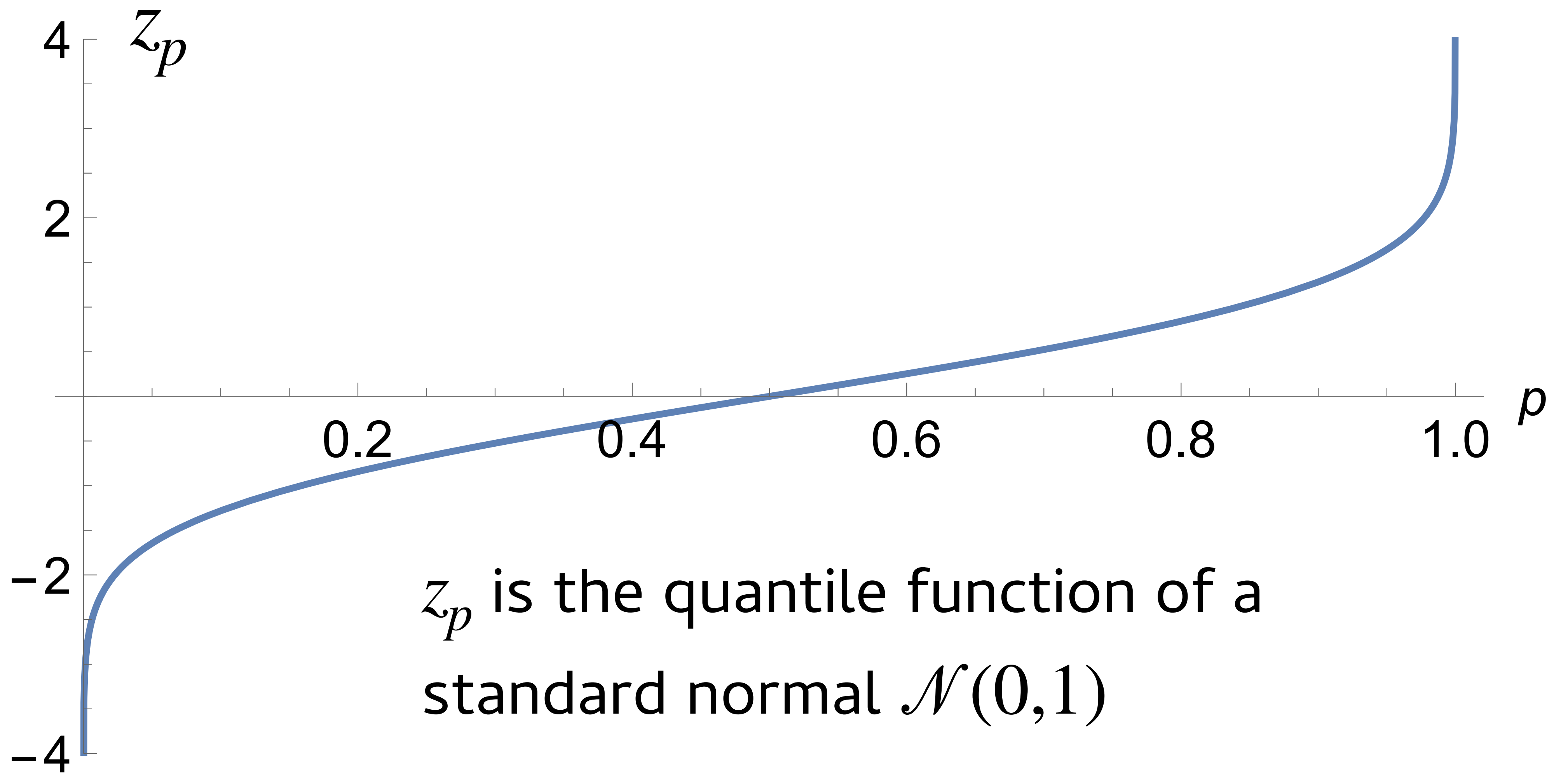
Cornish-Fisher Expansion

Cornish,E.A. & Fisher,R.A.; Revue De L'institut International De Statistique (1937)

$$q_p \sim m_1 + \sqrt{m_2}w$$

$$w = z_p + (z_p^2 - 1)\frac{m_3}{6} + (z_p^3 - 3z_p)\frac{m_4}{24} - (2z_p^3 - 5z_p)\frac{m_3^2}{36}$$



z_p is the quantile function of a standard normal $\mathcal{N}(0,1)$

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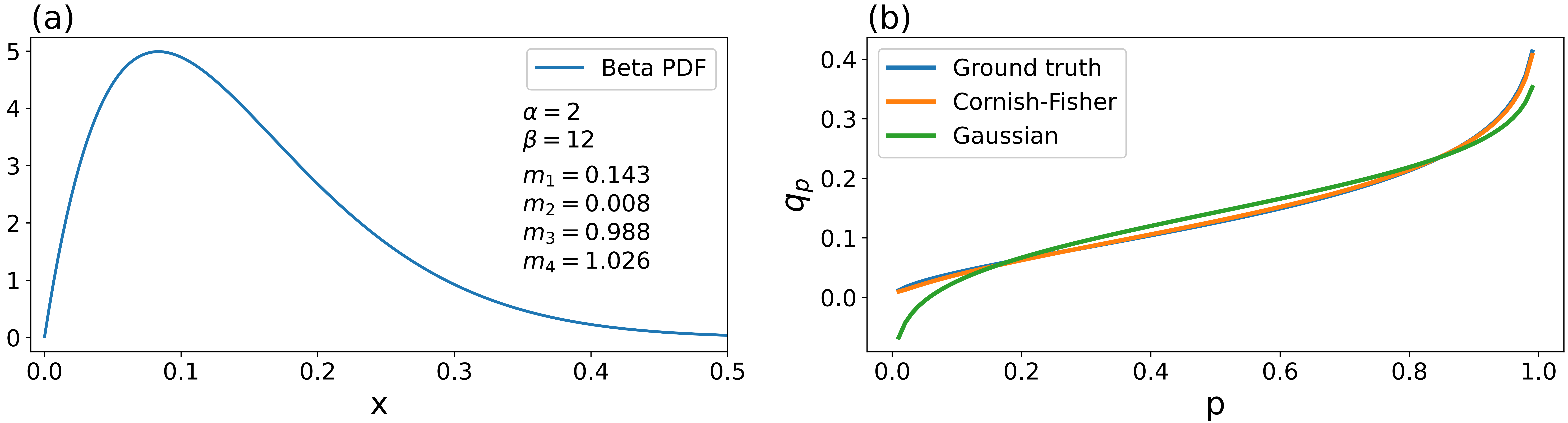

Exploring changes in quantiles and moments

(i.e., Median: $q_{p=0.5}$)

$$q_p(t) = q_p(m_1(t), m_2(t), m_3(t), m_4(t))$$

We focus on linear changes:

$$\frac{dq_p}{dt} = \frac{\partial q_p}{\partial m_1}\frac{dm_1}{dt} + \frac{\partial q_p}{\partial m_2}\frac{dm_2}{dt} + \frac{\partial q_p}{\partial m_3}\frac{dm_3}{dt} + \frac{\partial q_p}{\partial m_4}\frac{dm_4}{dt} = \sum_{i=1}^{4}\frac{\partial q_p}{\partial m_i}\frac{dm_i}{dt}$$

$$\frac{\partial q_p}{\partial m_1} = 1$$

$$\frac{\partial q_p}{\partial m_2} = \frac{1}{2}\frac{1}{\sqrt{m_2}}[z_p + \frac{1}{6}(z_p^2 - 1)\,m_3 - \frac{1}{36}(2z_p^3 - 5z_p)\,m_3^2 + \frac{1}{24}(z_p^3 - 3z_p)\,m_4]$$

$$\frac{\partial q_p}{\partial m_3} = \sqrt{m_2}\,[\frac{1}{6}(z_p^2 - 1) - \frac{1}{18}(2z_p^3 - 5z_p)\,m_3]$$

$$\frac{\partial q_p}{\partial m_4} = \frac{\sqrt{m_2}}{24}(z_p^3 - 3z_p)$$

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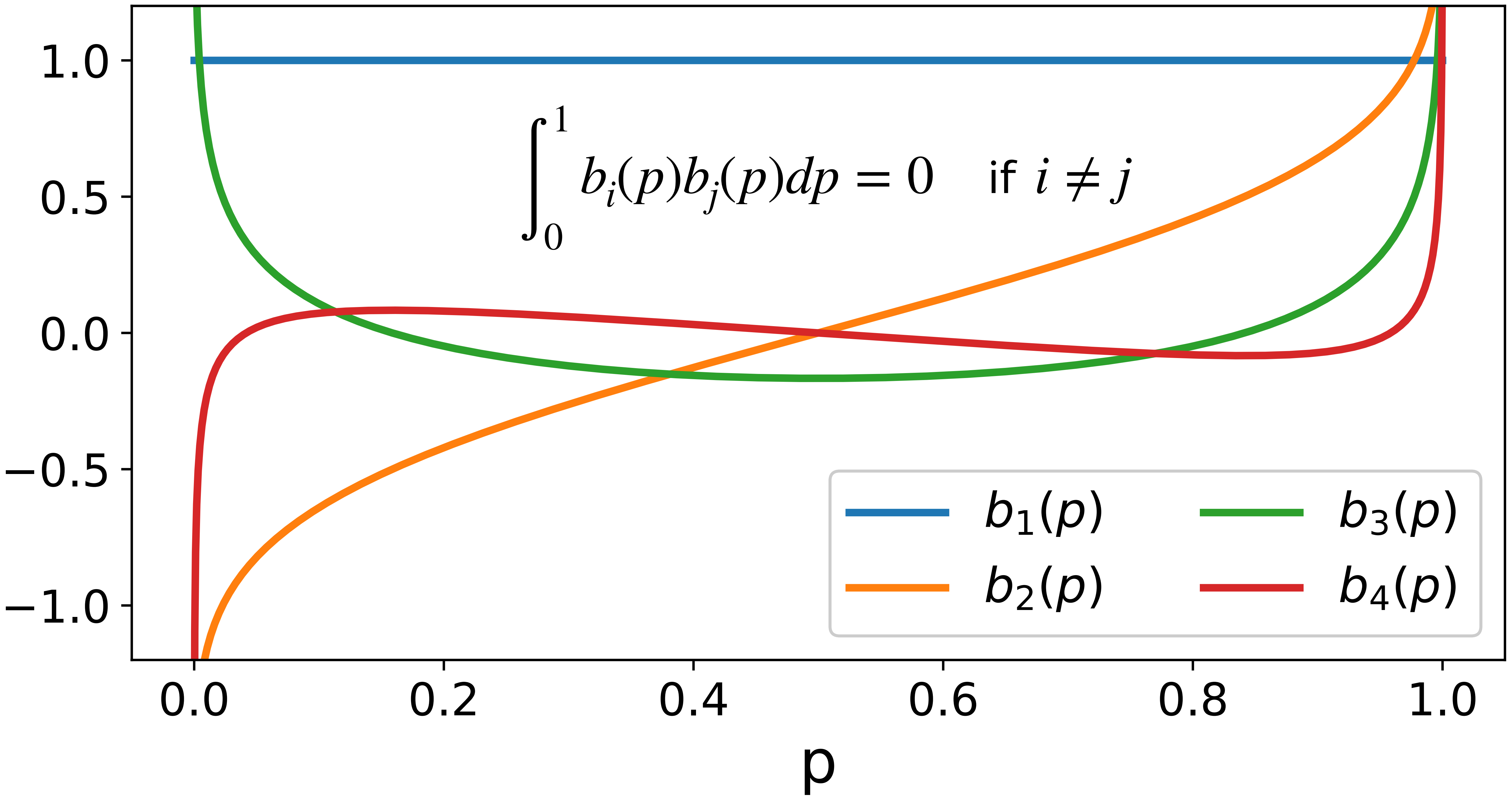
We assume small deviation from Gaussianity and evaluate each $\frac{\partial q_p}{\partial m_i}$ locally in $m_* = (m_1 = 0, m_2 = 1, m_3 = 0, m_4 = 0)$

$$b_1(p) = \frac{\partial q_p}{\partial m_1}\Big|_{m_*} = 1$$

$$b_2(p) = \frac{\partial q_p}{\partial m_2}\Big|_{m_*} = \frac{z_p}{2}$$

$$b_3(p) = \frac{\partial q_p}{\partial m_3}\Big|_{m_*} = \frac{1}{6}(z_p^2 - 1)$$

$$b_4(p) = \frac{\partial q_p}{\partial m_4}\Big|_{m_*} = \frac{1}{24}(z_p^3 - 3z_p)$$



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We assume small deviation from Gaussianity and evaluate each $\frac{\partial q_p}{\partial m_i}$ locally in $m_* = (m_1 = 0, m_2 = 1, m_3 = 0, m_4 = 0)$

$$\beta_1(q_p) \sim \left.\frac{dq_p}{dt}\right|_{m_*} = \sum_{i=1}^{4}\frac{dm_i}{dt}\left.\frac{\partial q_p}{\partial m_i}\right|_{m_*} = \sum_{i=1}^{4}\frac{dm_i}{dt}b_i(p)$$