# FastIsostasy.jl – A regional, 2.5D model for accelerated computation of glacial isostatic adjustment

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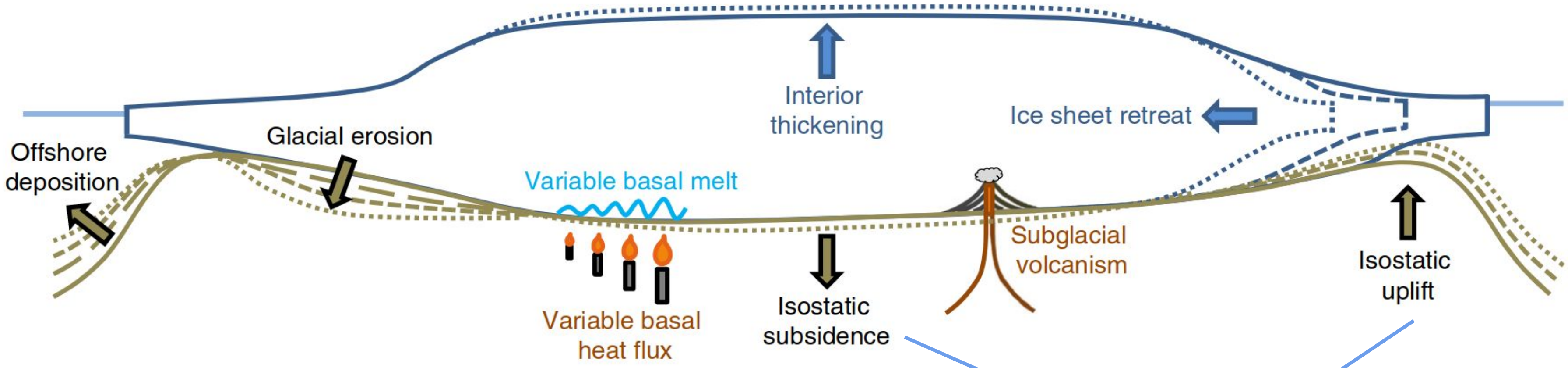
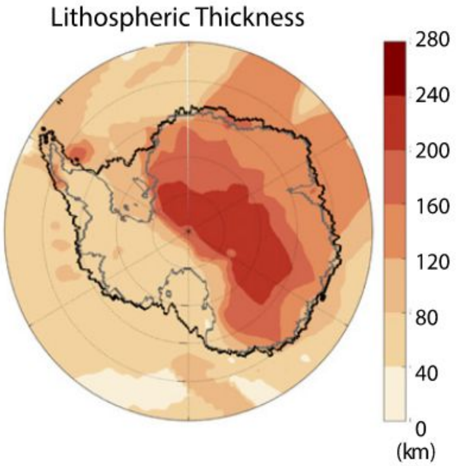
# Ice sheets and solid-Earth interact (a lot)



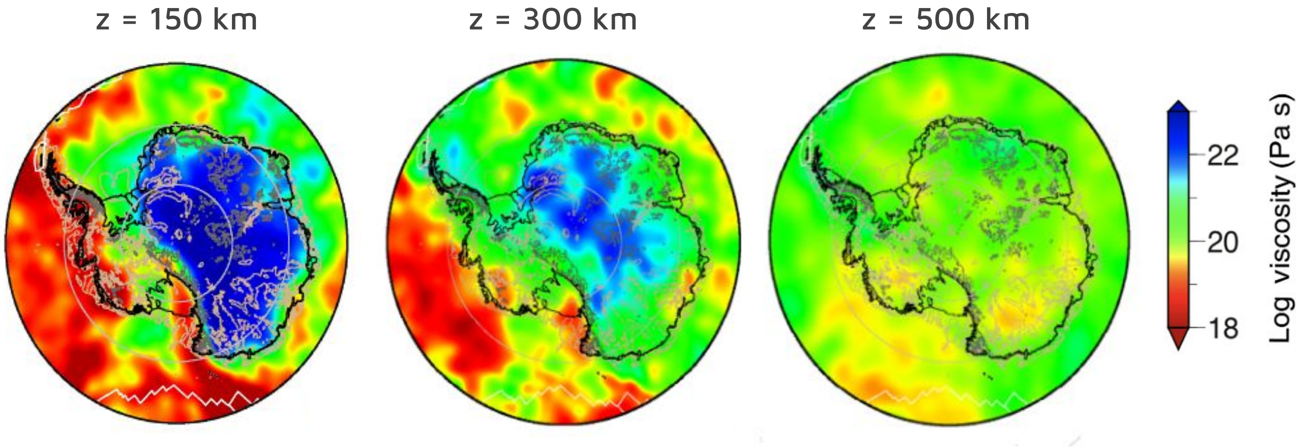

Whitehouse et al. (2019)

**Glacial isostatic adjustment (GIA)**

# Lithospheric thickness and upper-mantle viscosity: show large lateral variability (LV) over Antarctica




Gomez et al. (2018)




Ivins et al. (2022)

# Available regional models

**ELRA** = Elastic Lithosphere/Relaxed Asthenosphere:

Rigidity $D = ET^3/[12(1-\nu^2)]$

$$\rho_r g w + D\nabla^4 w = \sigma_{zz}$$

Asthenosphere density — Equilibrium displacement — Vertical load

Le Meur and Huybrecht (1996)

Transient displacement

$$\frac{\partial u}{\partial t} = -\frac{u - w}{\tau}$$

Relaxation time

# Available regional models

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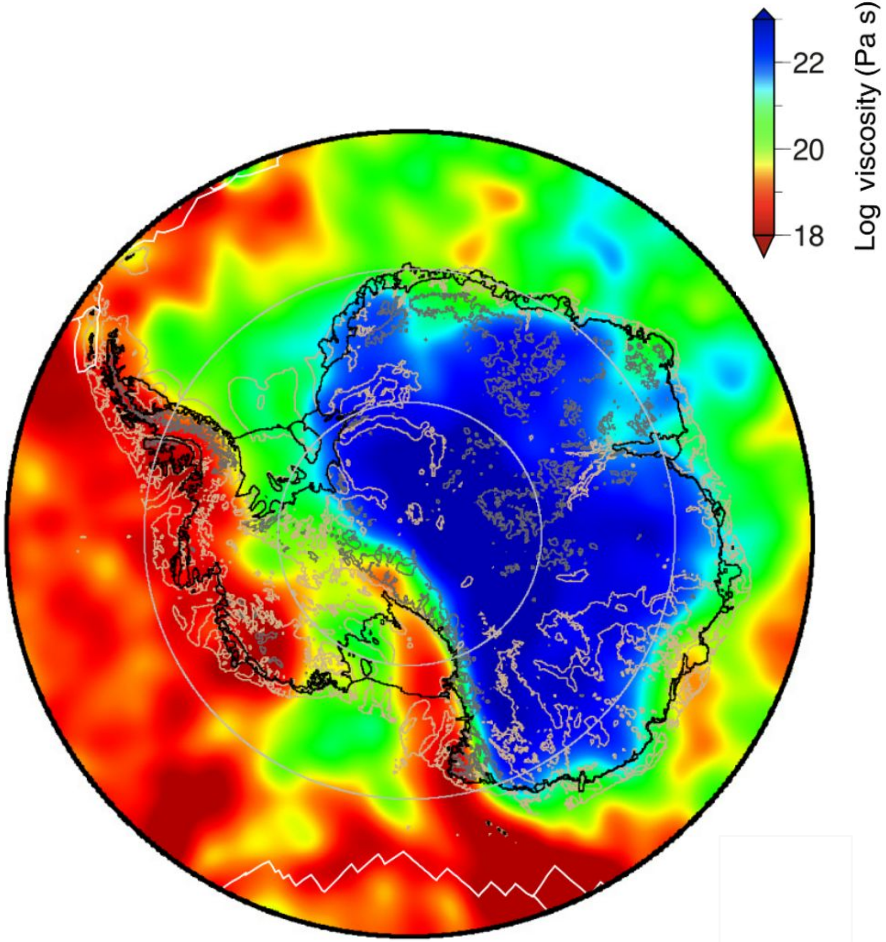
$$\rho_r g w + D\nabla^4 w = \sigma_{zz}$$

$$\frac{\partial u}{\partial t} = -\frac{u - w}{\tau}$$




Ivins et al. (2022)

# Available regional models

**ELRA** = Elastic Lithosphere/Relaxed Asthenosphere:

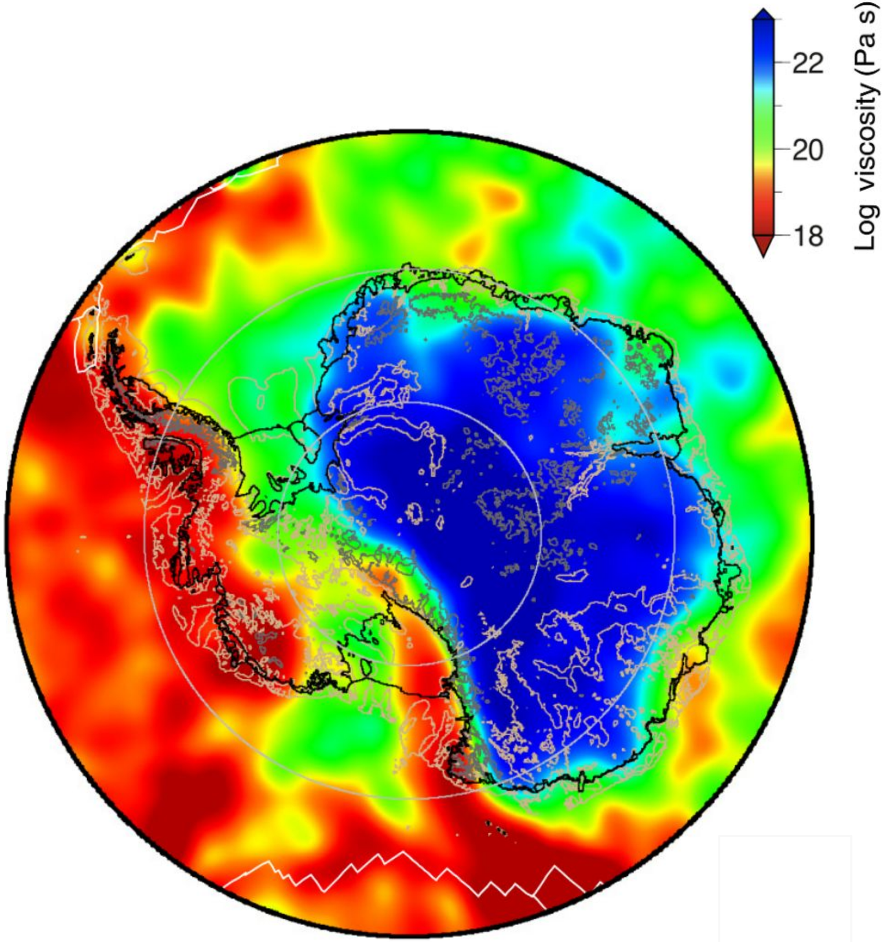
$$\rho_r g w + D\nabla^4 w = \sigma_{zz}$$

$$\frac{\partial u}{\partial t} = -\frac{u - w}{\tau}$$

**ELVA** = Elastic Lithosphere/Viscous Asthenosphere:

$$2\eta\,|\nabla|\,\frac{\partial u}{\partial t} + \rho_r g u + D\nabla^4 u = \sigma_{zz}$$

Cathles (1975), Lingle and Clark, Bueler et al. (2006)




Ivins et al. (2022)

# Available regional models

**ELRA** = Elastic Lithosphere/Relaxed Asthenosphere:

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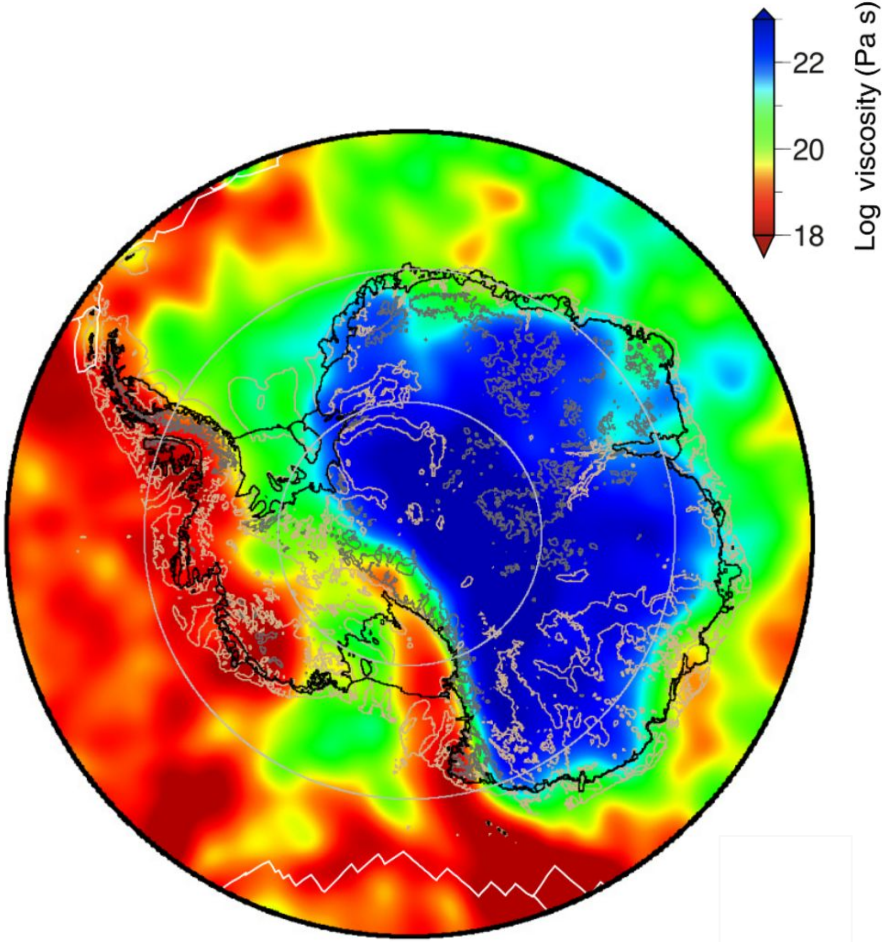
**ELVA** = Elastic Lithosphere/Viscous Asthenosphere:

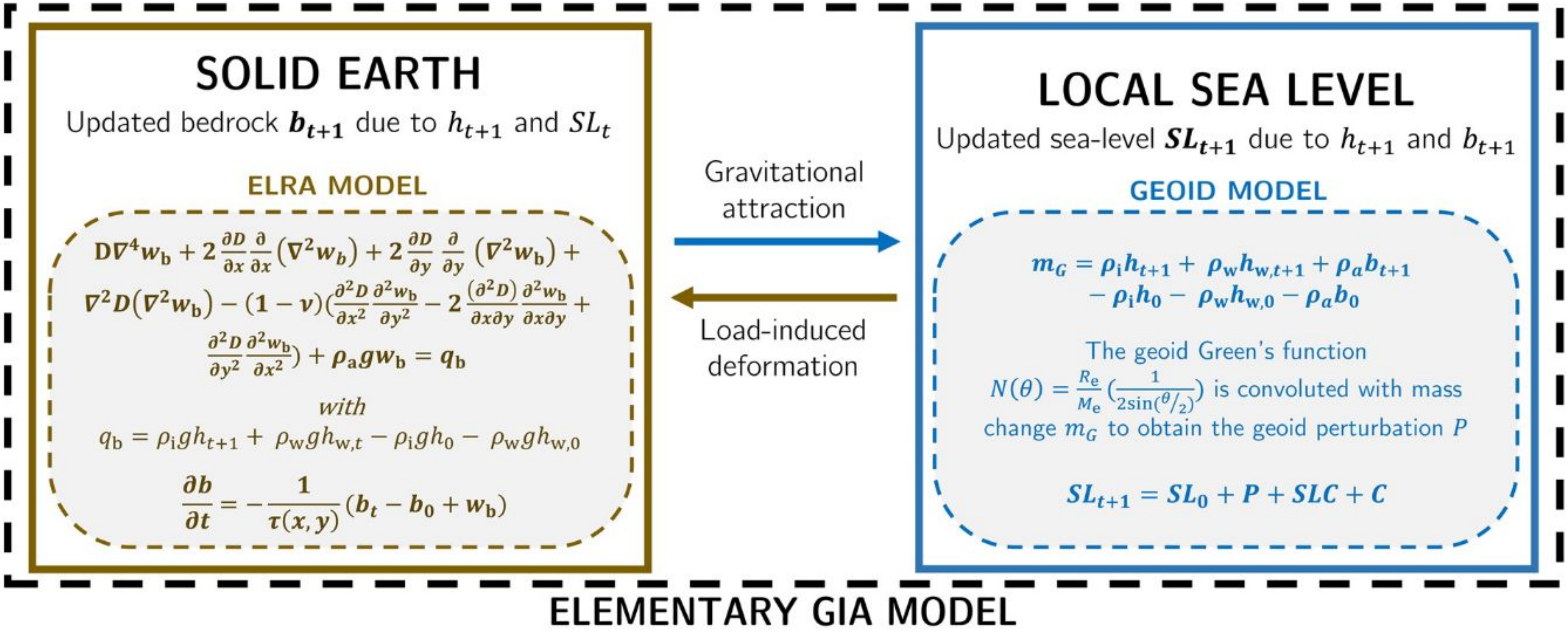
$$2\eta\,|\nabla|\,\frac{\partial u}{\partial t} + \rho_r g u + D\nabla^4 u = \sigma_{zz}$$

**No lateral variability (LV) of solid-Earth parameters!**



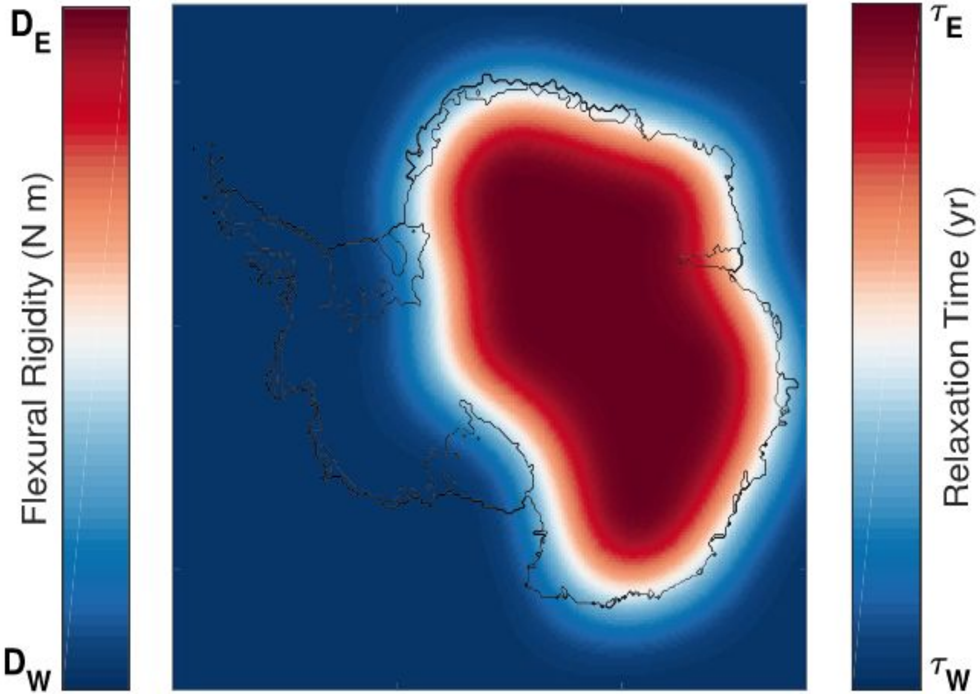
Ivins et al. (2022)

# Available regional models




Coulon et al. (2021)

# Available regional models




Coulon et al. (2021)

# LV: large influence on modelling of Antarctic Ice Sheet


1. A et al. (2013): Computations of the viscoelastic response of a 3-D compressible Earth to surface loading: an application to Glacial Isostatic Adjustment in Antarctica and Canada.
2. Austermann et al. (2021): The effect of lateral variations in Earth structure on Last Interglacial sea level.
3. Blank et al. (2021): Effect of Lateral and Stress-Dependent Viscosity Variations on GIA Induced Uplift Rates in the Amundsen Sea Embayment.
4. Coulon et al. (2021): Contrasting Response of West and East Antarctic Ice Sheets to Glacial Isostatic Adjustment.
5. Gomez et al. (2018): A Coupled Ice Sheet–Sea Level Model Incorporating 3D Earth Structure: Variations in Antarctica during the Last Deglacial Retreat.
6. Kaufmann et al. (2005): Lateral viscosity variations beneath Antarctica and their implications on regional rebound motions and seismotectonics.

# LV: large influence on modelling of Antarctic Ice Sheet

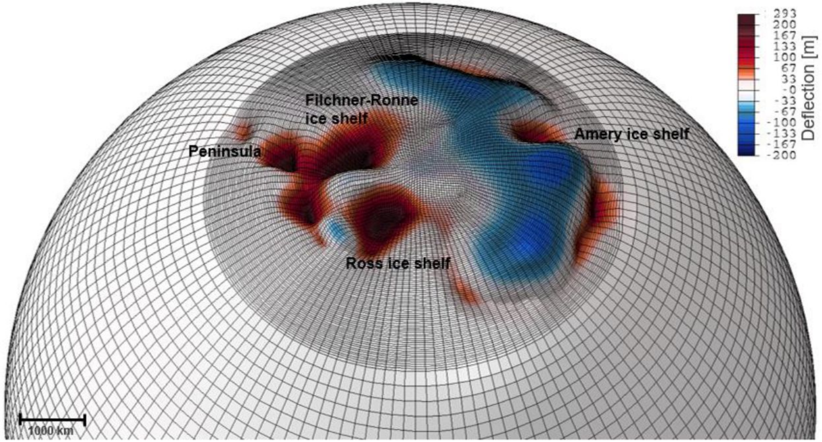

7. Konrad et al. (2014, 2015, 2016): The Deformational Response of a Viscoelastic Solid Earth Model Coupled to a Thermomechanical Ice Sheet Model; Potential of the solid-Earth response for limiting long-term West Antarctic Ice Sheet retreat in a warming climate; Sensitivity of Grounding-Line Dynamics to Viscoelastic Deformation of the Solid-Earth in an Idealized Scenario.

10. Nield et al. (2018): The impact of lateral variations in lithospheric thickness on glacial isostatic adjustment in West Antarctica.

11. Pollard et al. (2017): Variations of the Antarctic Ice Sheet in a Coupled Ice Sheet-Earth-Sea Level Model: Sensitivity to Viscoelastic Earth Properties.

12. Spada et al. (2006): Variations of the Antarctic Ice Sheet in a Coupled Ice Sheet-Earth-Sea Level Model: Sensitivity to Viscoelastic Earth Properties.

13. Van Calcar et al. (preprint): Simulation of a fully coupled 3D GIA - ice-sheet model for the Antarctic Ice Sheet over a glacial cycle.


Van Calcar et al. (preprint)

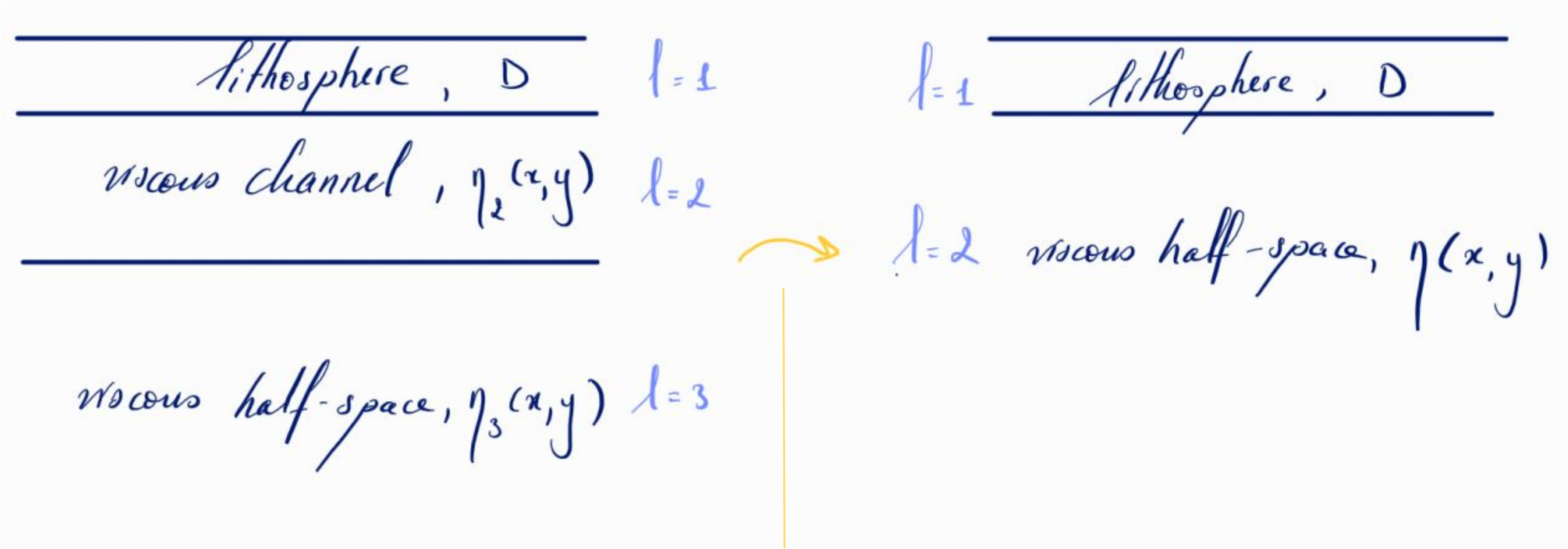
# Available regional models...

| | ELRA | ELVA | LV-ELRA | FastIsostasy |
|---|---|---|---|---|
| **Explicit viscosity** | × | ✓ | × | ✓ |
| **Lateral variability** | × | × | ✓ × | ✓ |
| **Computation domain** | regional | regional | regional | regional |
| **Numerical scheme** | FDM | FCM | FDM | FDM/FCM |
| **Computational cost** | low | very low | low-intermediate | low-intermediate |

LV-ELVA

# Description and results

# An n-layer model



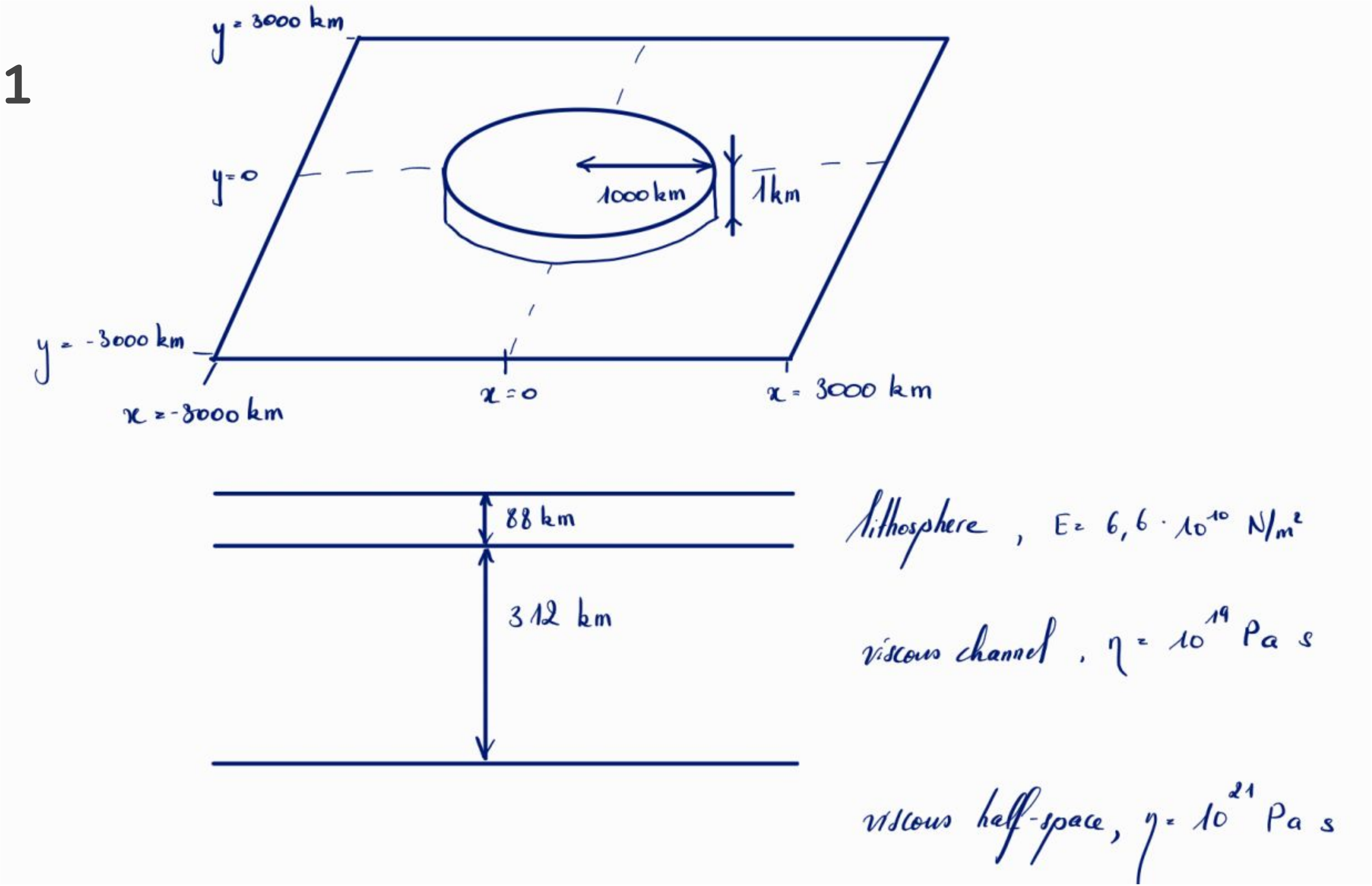

$$R(\kappa,\tilde{\eta}) = \frac{2\tilde{\eta}C(\kappa)S(\kappa) + \left(1-\tilde{\eta}^2\right)T_c^2\kappa^2 + \tilde{\eta}^2 S(\kappa)^2 + C(\kappa)^2}{\left(\tilde{\eta}+\tilde{\eta}^{-1}\right)C(\kappa)S(\kappa) + \left(\tilde{\eta}-\tilde{\eta}^{-1}\right)T_c\kappa + S(\kappa)^2 + C(\kappa)^2}$$

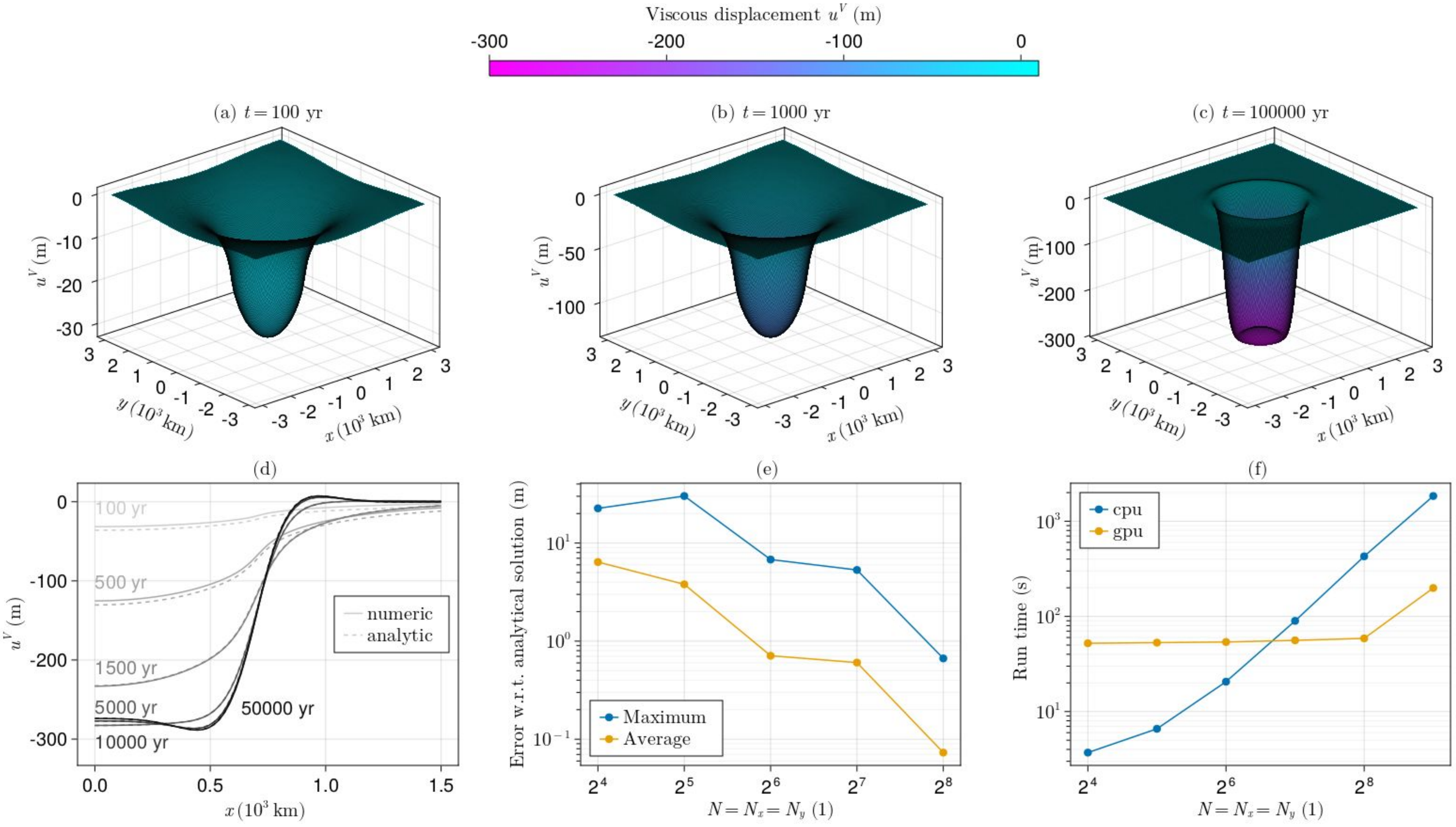
# Allow η = η(x, y, t), D = D(x,y)

$$|\nabla| \left( \frac{\partial(2\eta u^V)}{\partial t} \right) = \sigma_{zz} - \rho g u^V - D\nabla^4 u^V - 2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\left(\nabla^2 u^V\right) - 2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\left(\nabla^2 u^V\right) - \nabla^2 D\left(\nabla^2 u^V\right)$$
$$+(1-\nu)\left( \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 u^V}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 u^V}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 u^V}{\partial x^2} \right)$$

**Combine FDM with Fourier collocation method (FCM)!**
**Use Fast-Fourier Transform!**

# Test 1


y = 3000 km
y = 0
y = -3000 km
1000 km
1 km
x = -3000 km
x = 0
x = 3000 km
88 km
312 km
lithosphere , E = 6,6 · 10^10 N/m^2
viscous channel , η = 10^19 Pa s
viscous half-space, η = 10^21 Pa s



Viscous displacement $u^V$ (m)
-300
-200
-100
0
(a) $t = 100$ yr
(b) $t = 1000$ yr
(c) $t = 100000$ yr
$u^V$ (m)
$y\,(10^3$ km)
$x\,(10^3$ km)
(d)
100 yr
500 yr
1500 yr
5000 yr
10000 yr
50000 yr
numeric
analytic
$x\,(10^3$ km)
(e)
Error w.r.t. analytical solution (m)
Maximum
Average
$N = N_x = N_y$ (1)
(f)
Run time (s)
cpu
gpu
$N = N_x = N_y$ (1)

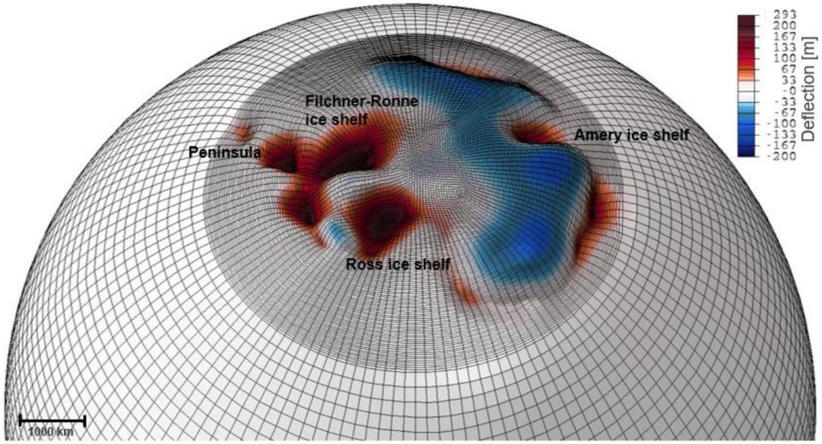
# Comparing things that do not compare...

Van Calcar et. al (preprint): 3D GIA model coupled to ice-sheet model

- Coupling time step: 500 years → iterative coupling
- Simulation time: 130 kyr
- 16 CPU → Computation time = 37 days
  (can be reduced to 5 days if non-iterative coupling)
- After talking with Caroline v.C.: fairer comparison
  Would be 15 hours of computation time

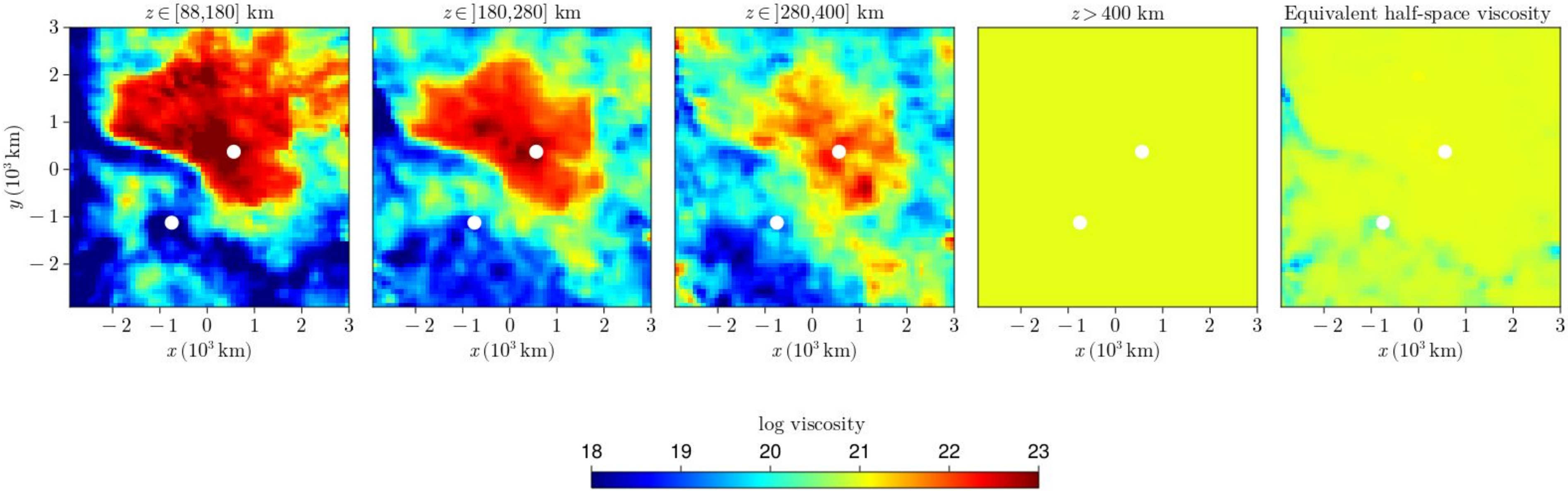
FastIsostasy

- Time step: 1 year
- Simulation time: 100 kyr
- 1 CPU → Computation time < 7 minutes
- 1 GPU → Computation time = 1 minute

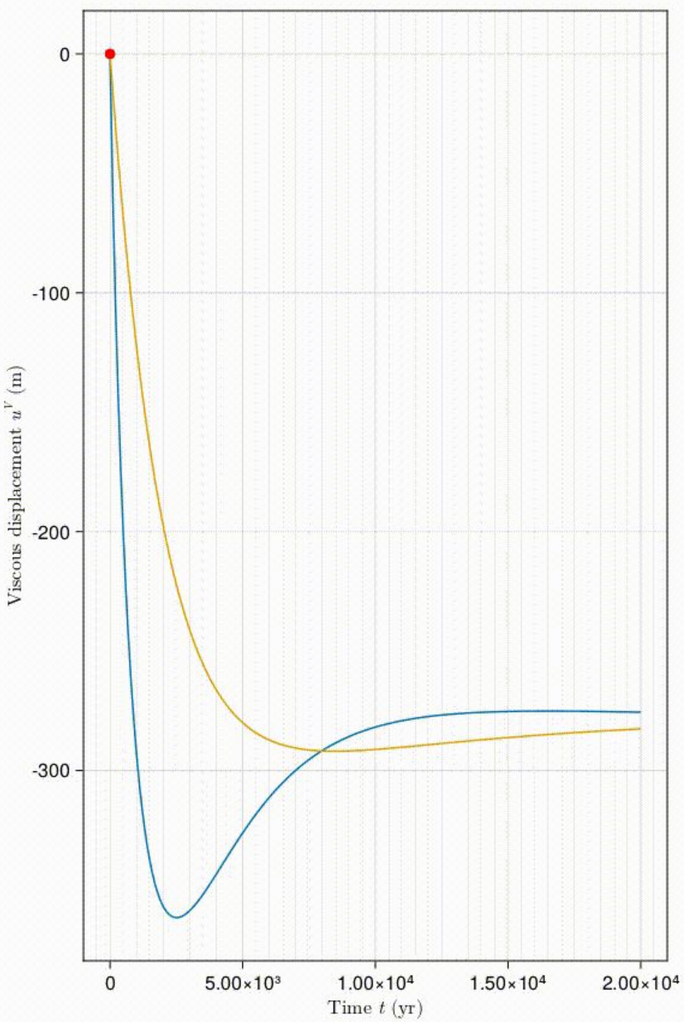


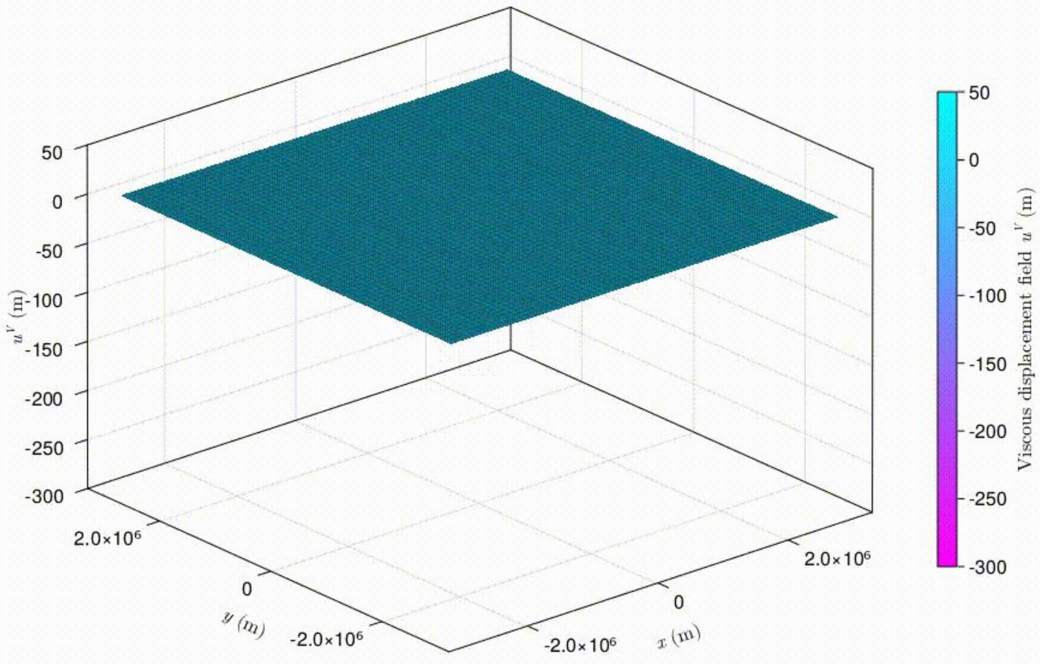

Van Calcar et al. (preprint)

# Test 4

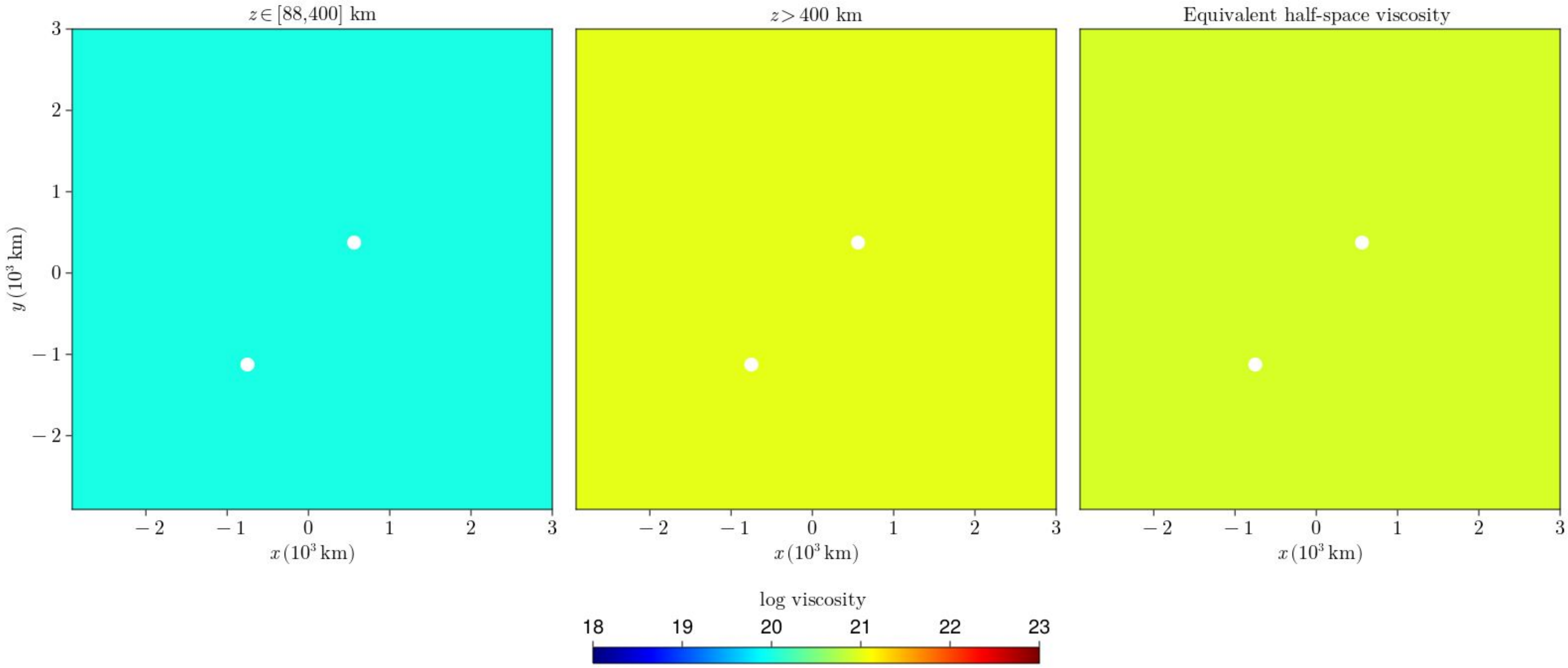



Based on Wiens et al. (2022)

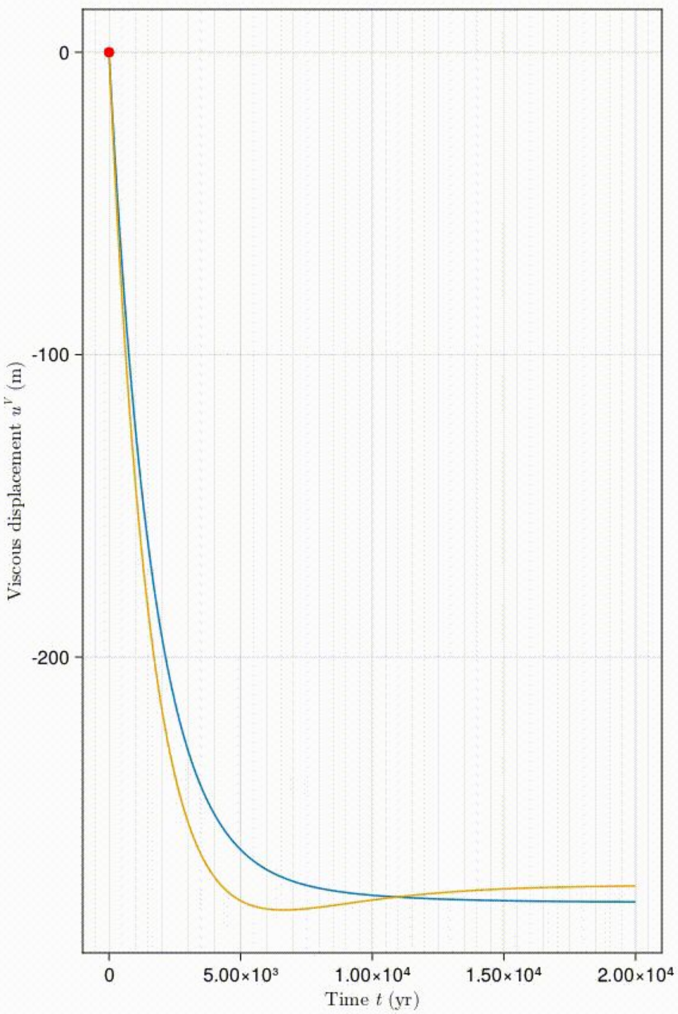

0
-100
-200
-300
0
5.00×10³
1.00×10⁴
1.50×10⁴
2.00×10⁴
Time t (yr)
Viscous displacement u^V (m)

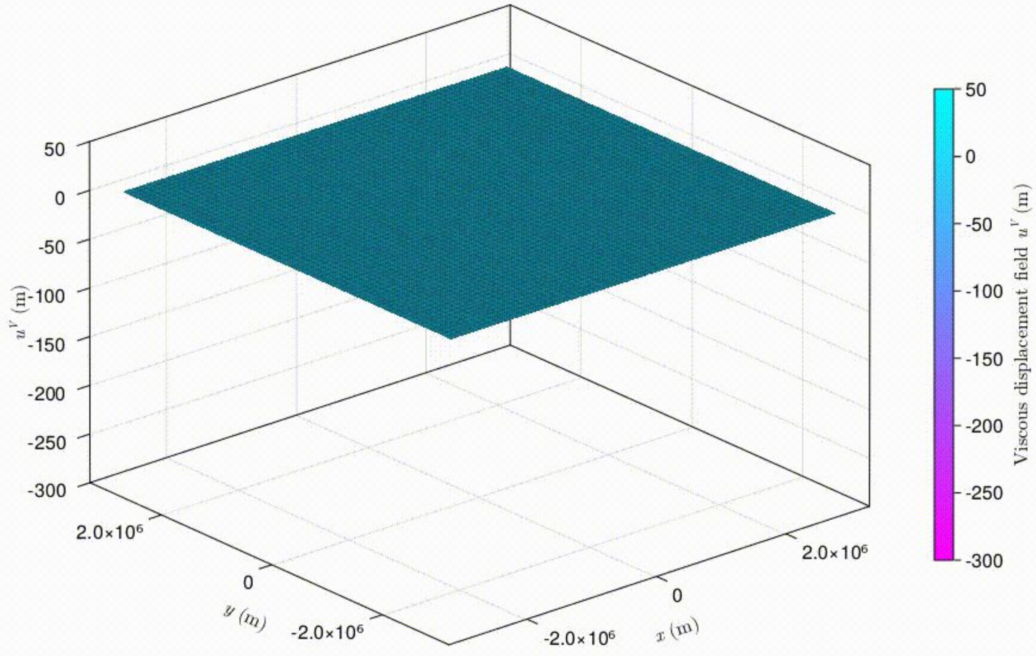

50
0
-50
-100
-150
-200
-250
-300
u^V (m)
2.0×10⁶
0
-2.0×10⁶
y (m)
-2.0×10⁶
0
2.0×10⁶
x (m)
Viscous displacement field u^V (m)

# Test 4

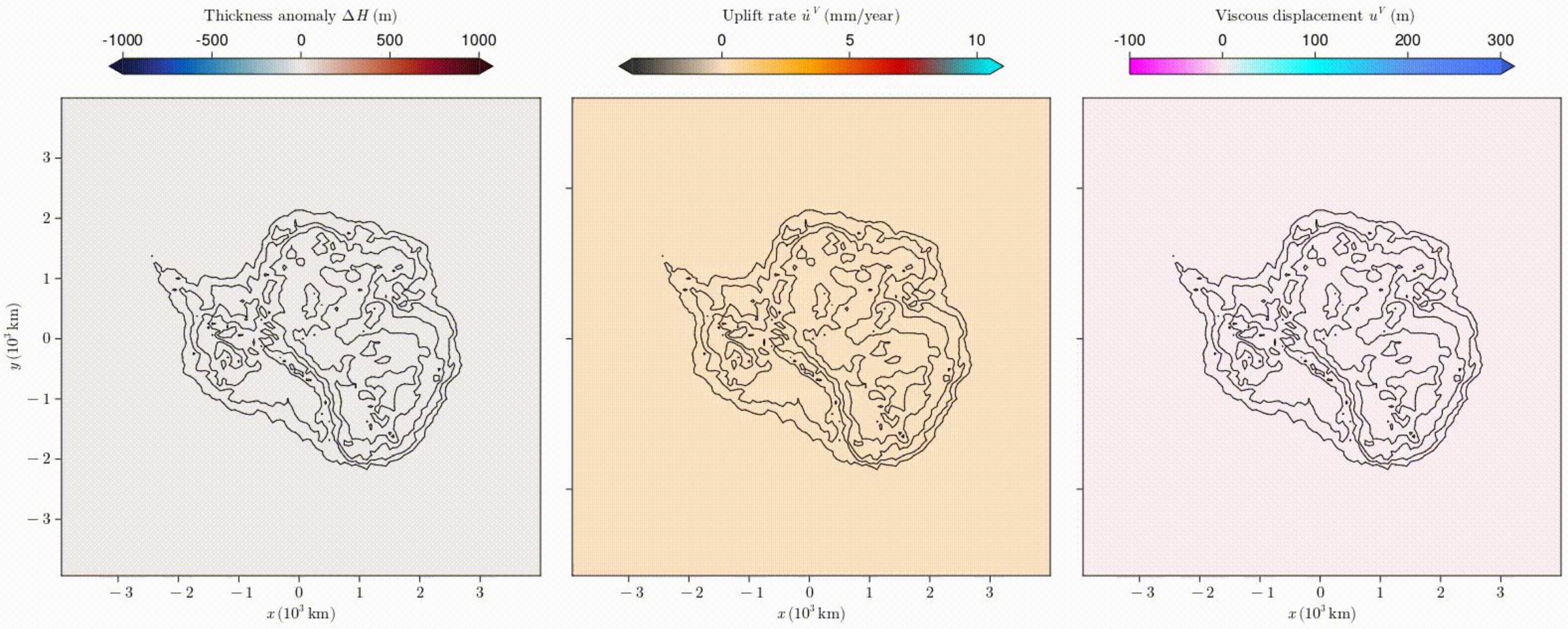


$z \in [88,400]$ km
$z > 400$ km
Equivalent half-space viscosity
$y\,(10^3\,\mathrm{km})$
$x\,(10^3\,\mathrm{km})$
$x\,(10^3\,\mathrm{km})$
$x\,(10^3\,\mathrm{km})$
3
2
1
0
−1
−2
−2
−1
0
1
2
3
log viscosity
18
19
20
21
22
23


0
-100
-200
Viscous displacement $u^V$ (m)
0
5.00×10³
1.00×10⁴
1.50×10⁴
2.00×10⁴
Time $t$ (yr)

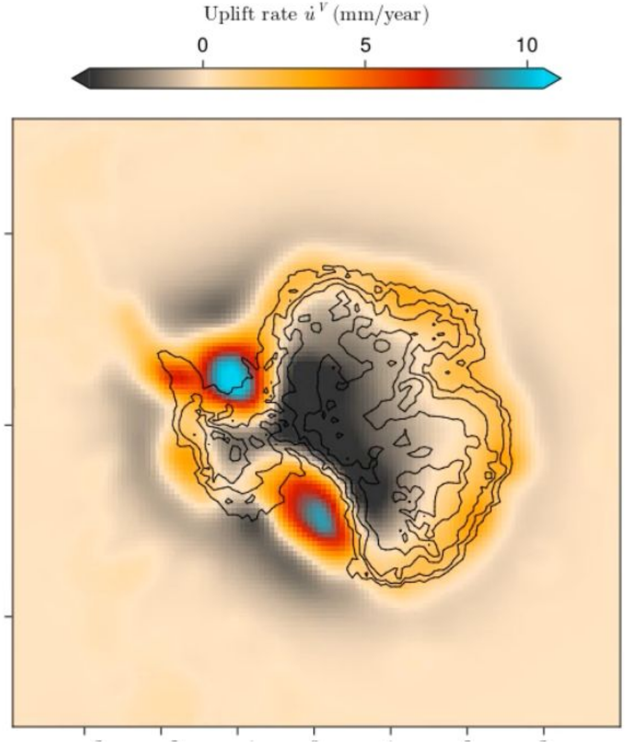

50
0
-50
-100
-150
-200
-250
-300
$u^V$ (m)
2.0×10⁶
0
-2.0×10⁶
$y$ (m)
-2.0×10⁶
0
2.0×10⁶
$x$ (m)
50
0
-50
-100
-150
-200
-250
-300
Viscous displacement field $u^V$ (m)

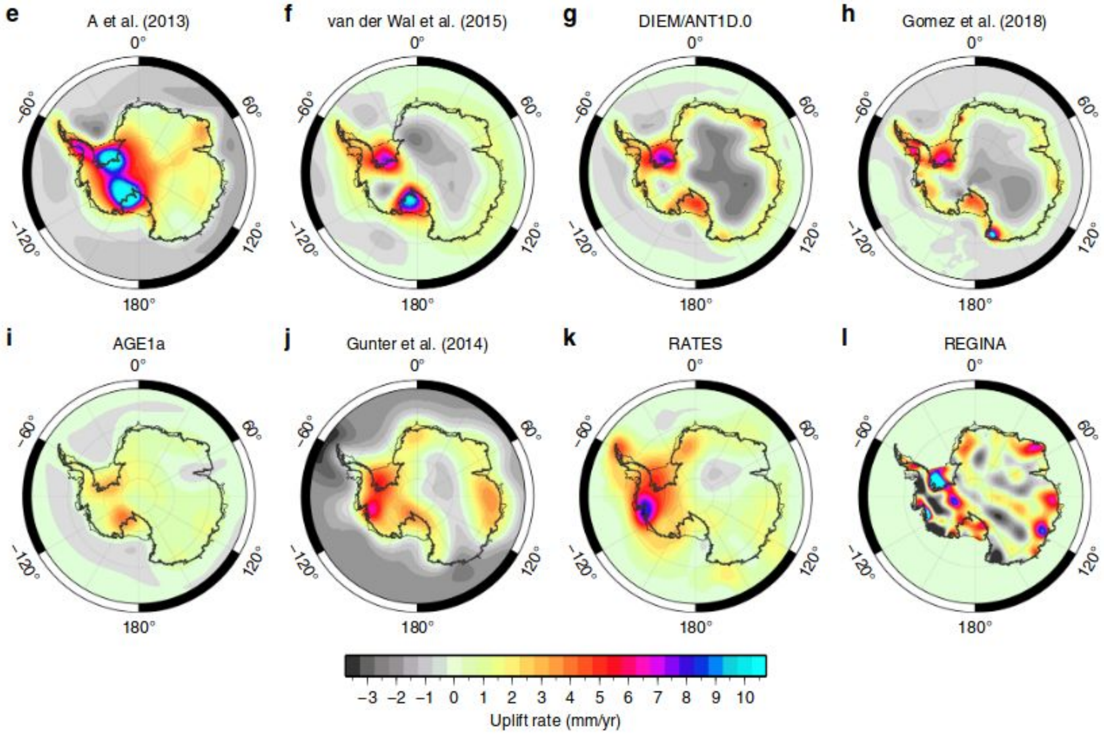
# Test 5



Thickness anomaly ΔH (m)
-1000
-500
0
500
1000
Uplift rate $\dot{u}^V$ (mm/year)
0
5
10
Viscous displacement $u^V$ (m)
-100
0
100
200
300
$y$ ($10^3$ km)
3
2
1
0
−1
−2
−3
$x$ ($10^3$ km)

# Test 5

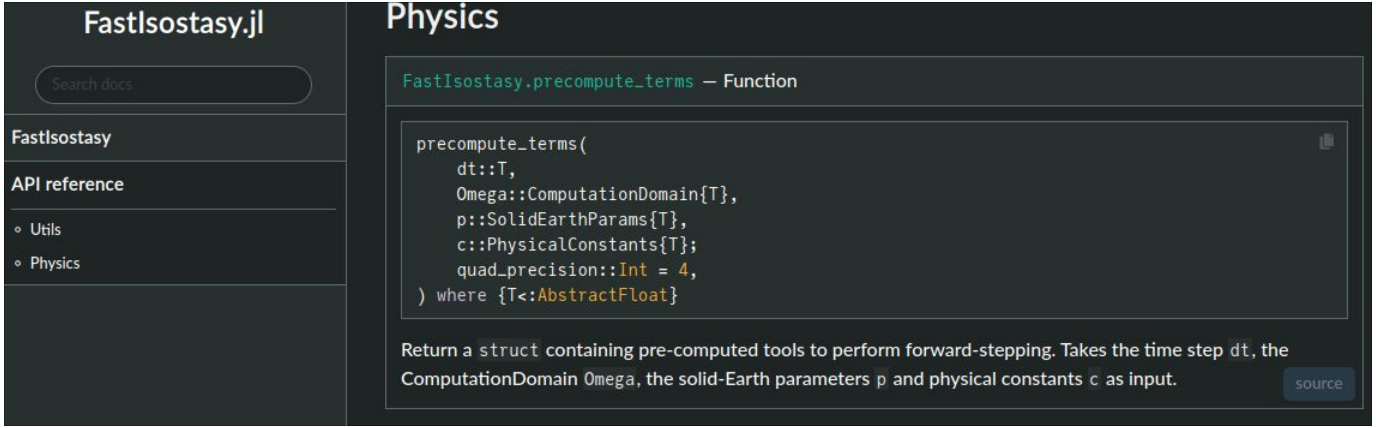



Whitehouse et al. (2019)

# Conclusions on FastIsostasy

- Only regional model to fully account for LV
- Extensively tested
- Holds the promise of being fast
- Can resolve the fast uplift of low-viscosity regions
- Avoids tedious coupling of ice-sheet model with 3D GIA

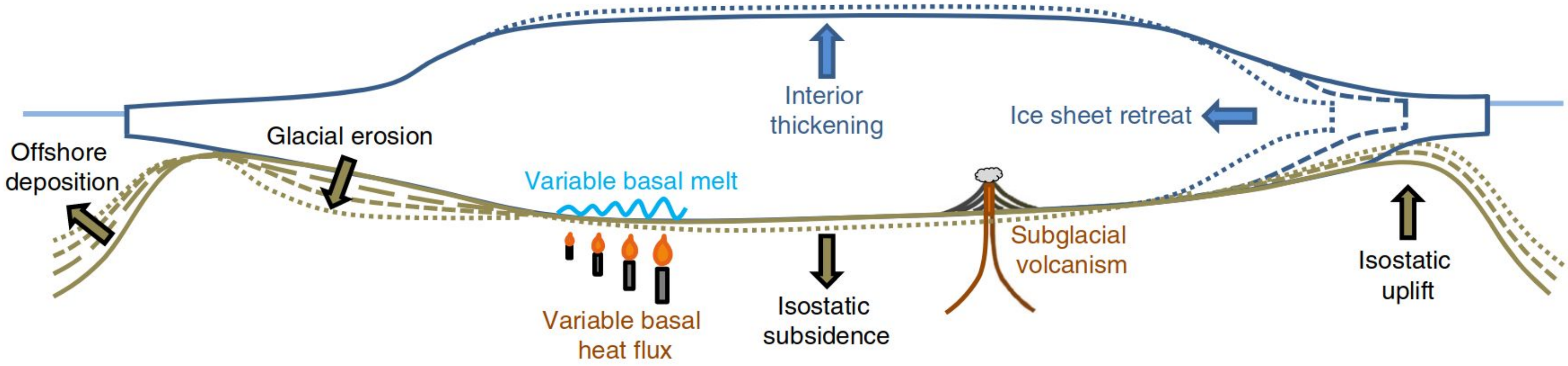


# Future work

- Implement geoid and sea-level equation
- Implement better elastic response
- Adaptive time stepping
- Higher-order methods for time integration
- Publish model!

# Appendix

# Ice sheets and solid-Earth interact (a lot)




Whitehouse et al. (2019)

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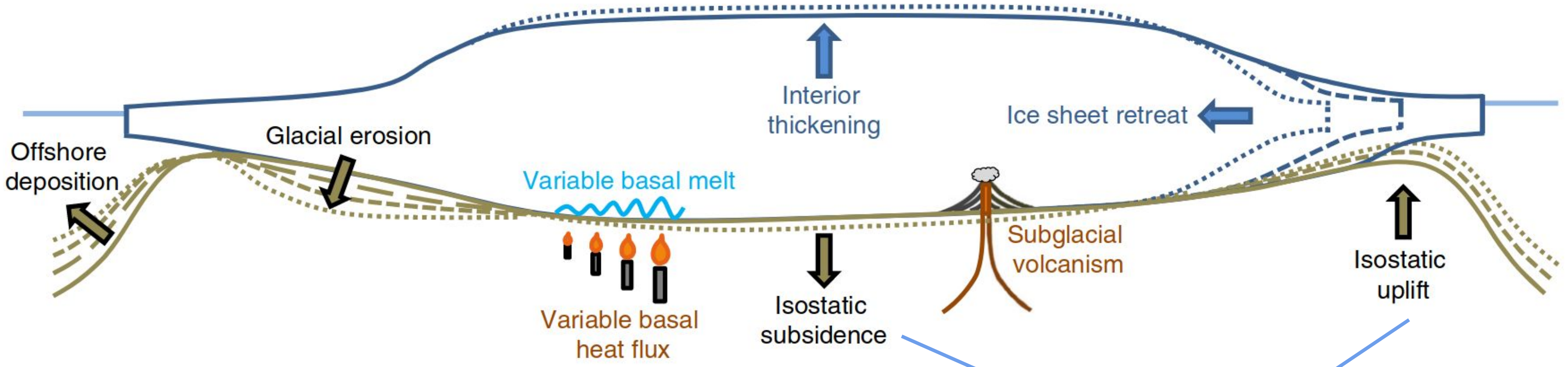
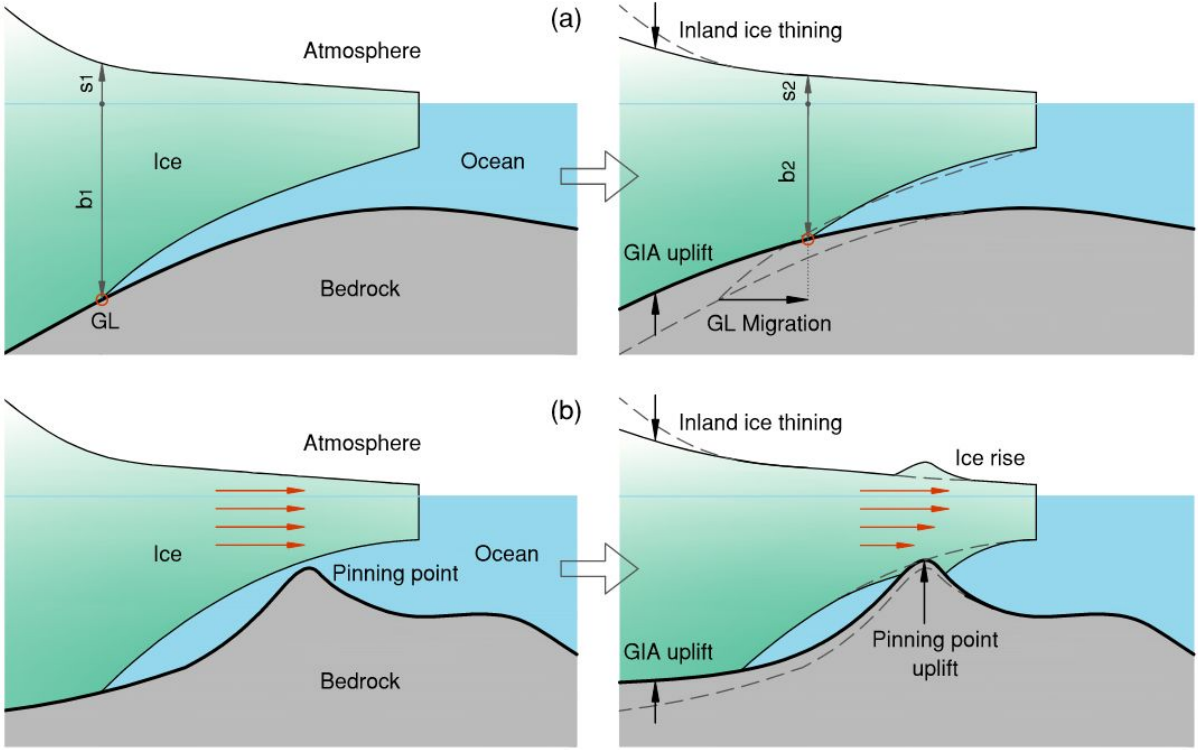



Whitehouse et al. (2019)

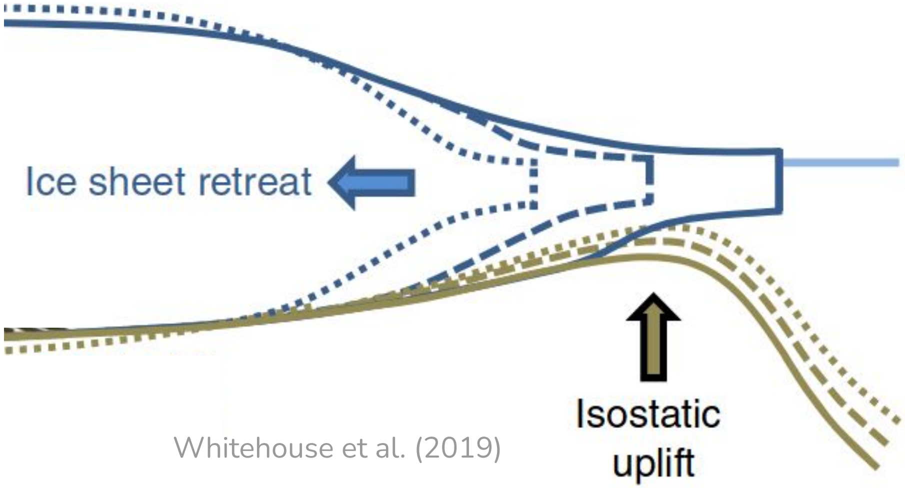
**Glacial isostatic adjustment (GIA)**

# GIA is particularly important for marine ice-sheets

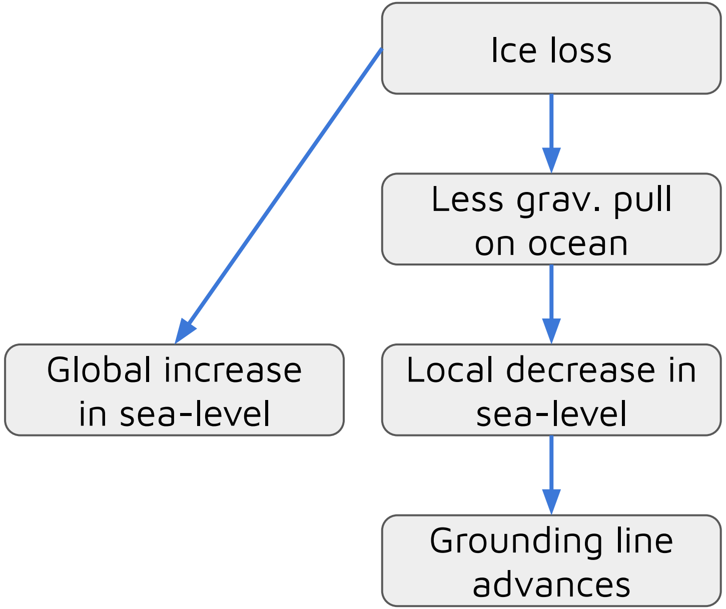



Adhikari et al. (2014)

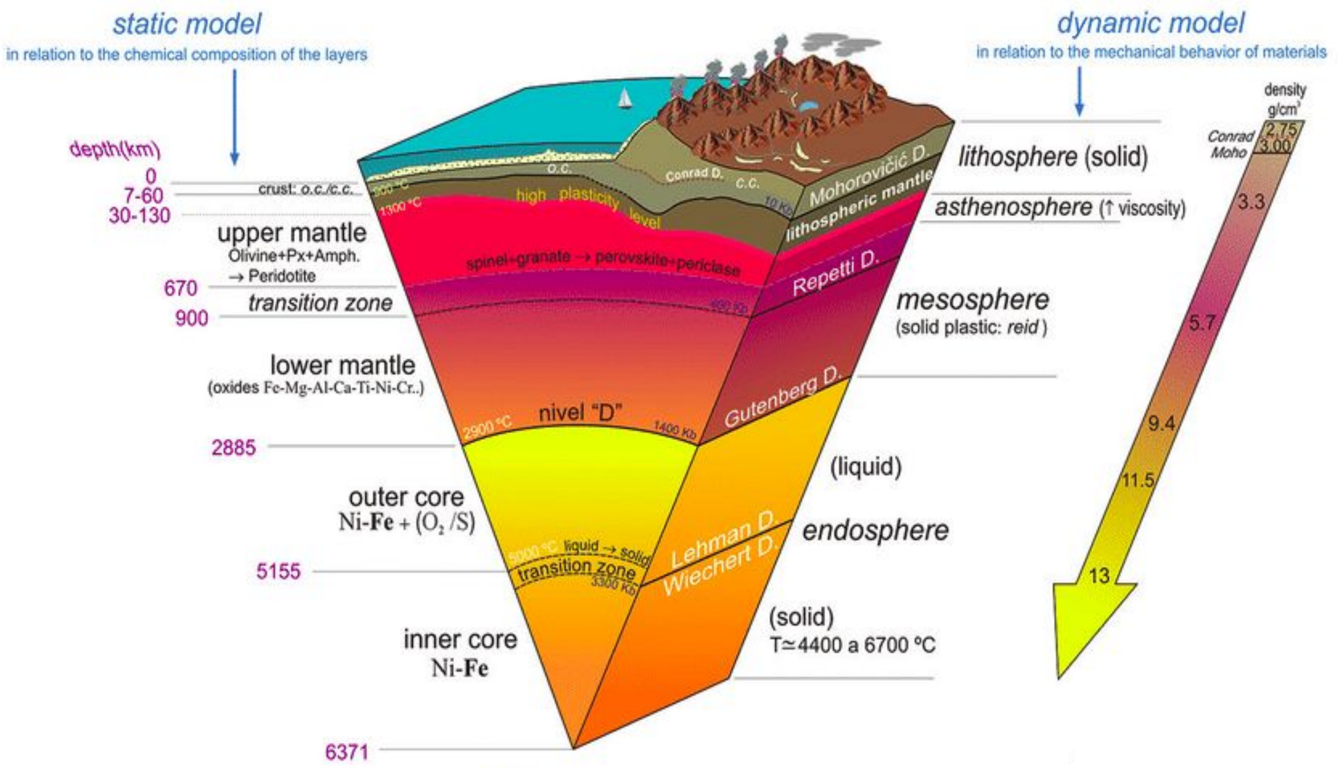
# GIA model typically contains sea-level equation




Whitehouse et al. (2019)

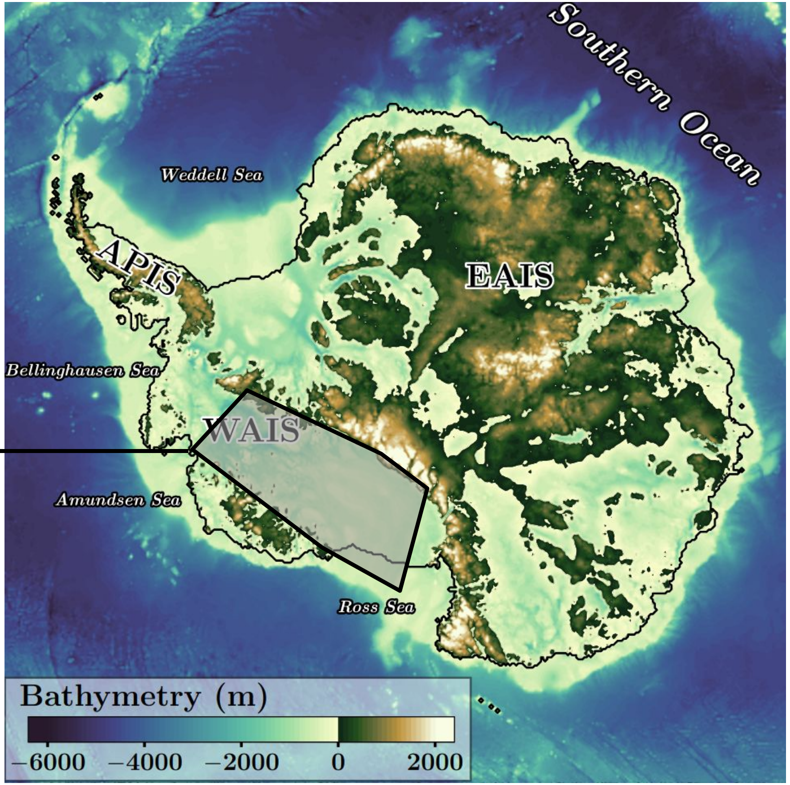


# Solid-Earth has layered structure



# East/West-Antarctica: (very) different geological nature

Rift system



# LV: large influence on modelling of Antarctic Ice Sheet

So far Yelmo + ELRA:

- relaxation time > 1000 years → R-tipping
- relaxation time < 1000 years → No R-tipping

# Combine FDM with Fourier collocation method (FCM)

**Fourier**: $$\mathcal{F}(|\nabla|\boldsymbol{u}) = \boldsymbol{\kappa} \circ \mathcal{F}(\boldsymbol{u})$$

**FDM**: $$\frac{\partial \boldsymbol{u}}{\partial x} = \mathcal{D}_{\Delta x}(\boldsymbol{u})$$

**Explicit Euler**: $$\frac{\partial \boldsymbol{u}}{\partial t} \simeq \frac{\boldsymbol{u}_{k+1} - \boldsymbol{u}_k}{\Delta t}$$

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**Problem:** $$|\nabla| \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \boldsymbol{u}}{\partial x}$$

**Approximation:** $$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + \Delta t \cdot \mathcal{F}^{-1}\left(\mathcal{F}\left(\mathcal{D}_{\Delta x}(\boldsymbol{u})\right) \oslash \boldsymbol{\kappa}\right)$$

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**Fourier:** $\mathcal{F}(|\nabla|\boldsymbol{u}) = \boldsymbol{\kappa} \circ \mathcal{F}(\boldsymbol{u})$

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**Problem:** $|\nabla| \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \boldsymbol{u}}{\partial x}$

**Approximation:** $\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + \Delta t \cdot \underline{\mathcal{F}^{-1}\left(\mathcal{F}\left(\mathcal{D}_{\Delta x}\right.\right.}(\boldsymbol{u})) \oslash \boldsymbol{\kappa})$

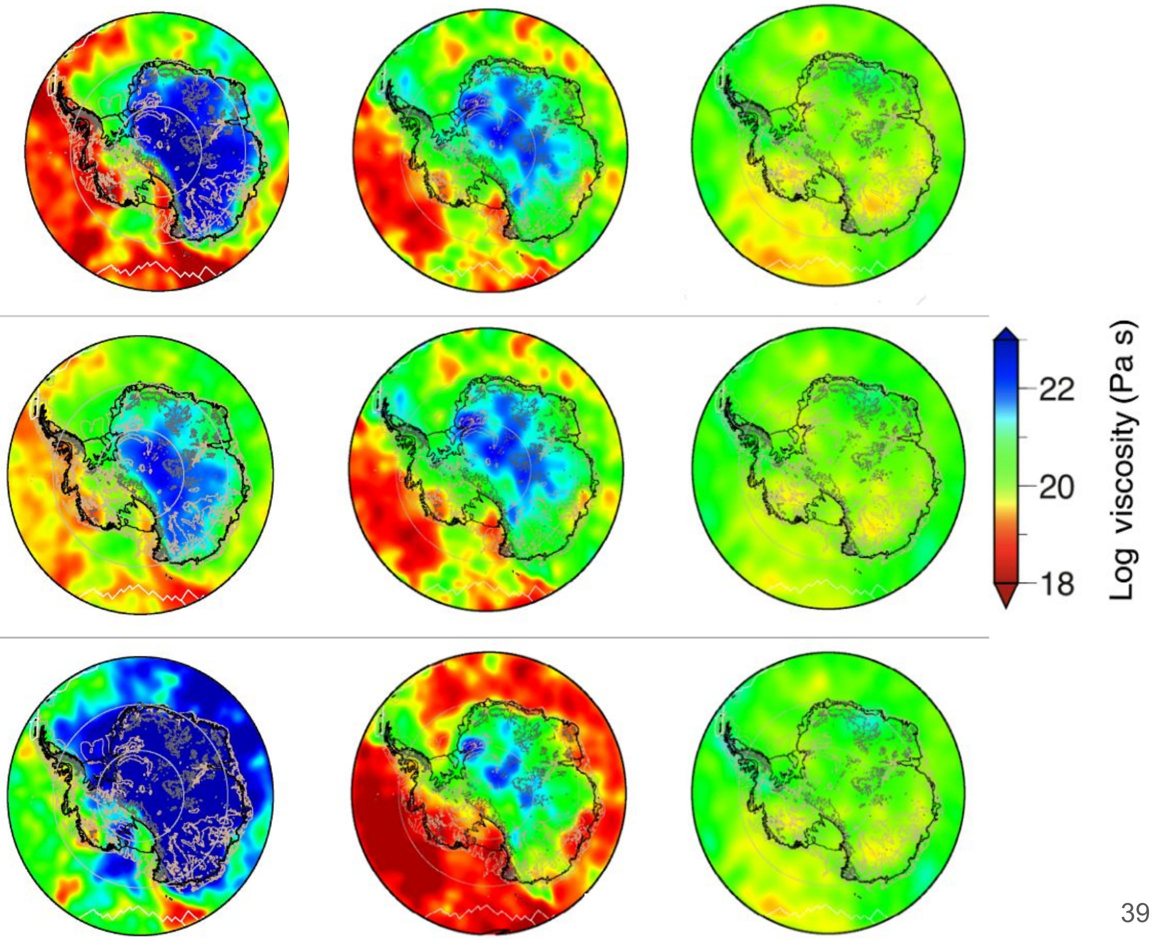
**Cheap thanks to FFT!**

z = 150 km | z = 300 km | z = 500 km

## Estimation method 1

## Estimation method 2

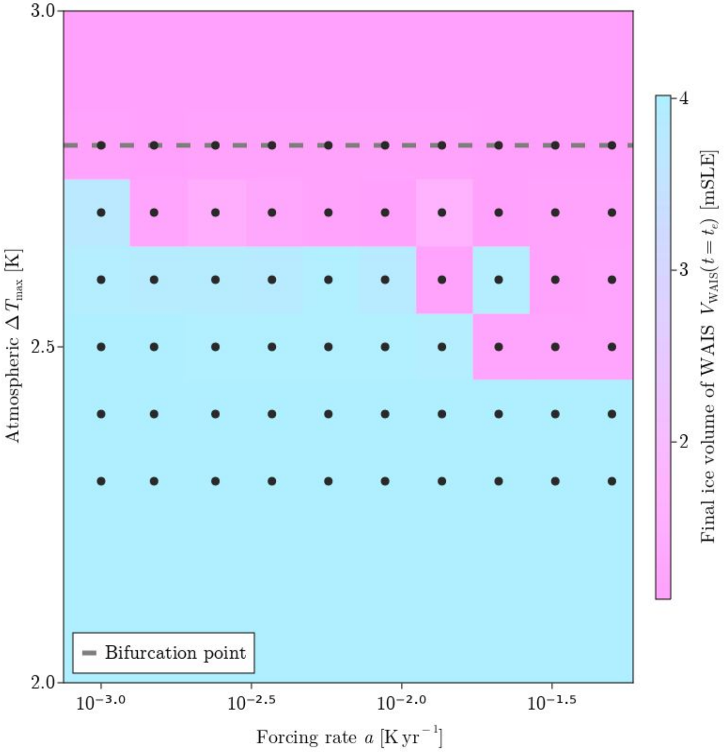
## Estimation method 3




Ivins et al. (2022)

# What does it have to do with tipping?

ELRA: τ = 3000 yr



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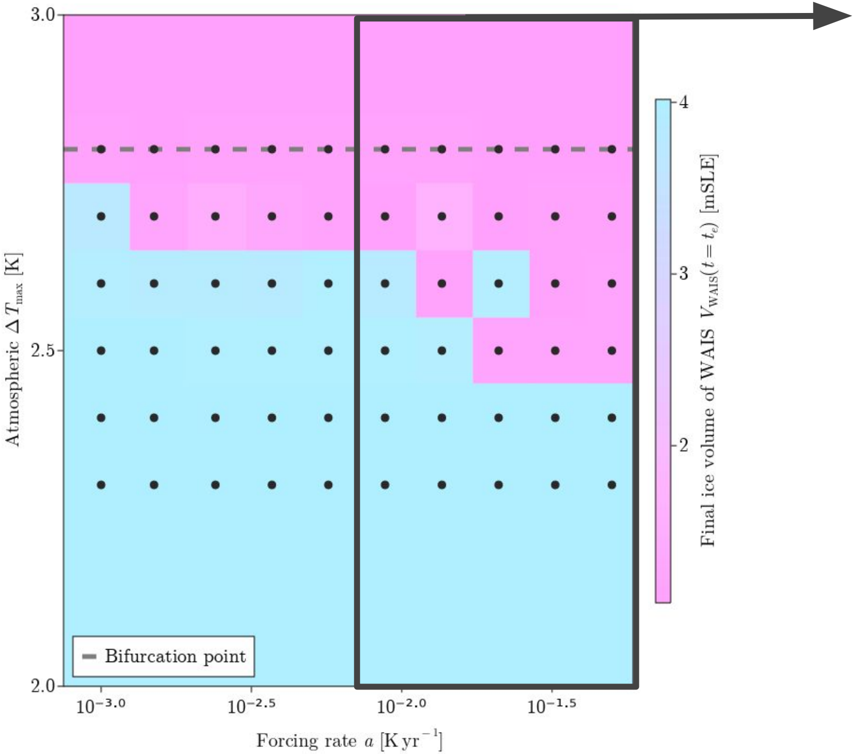
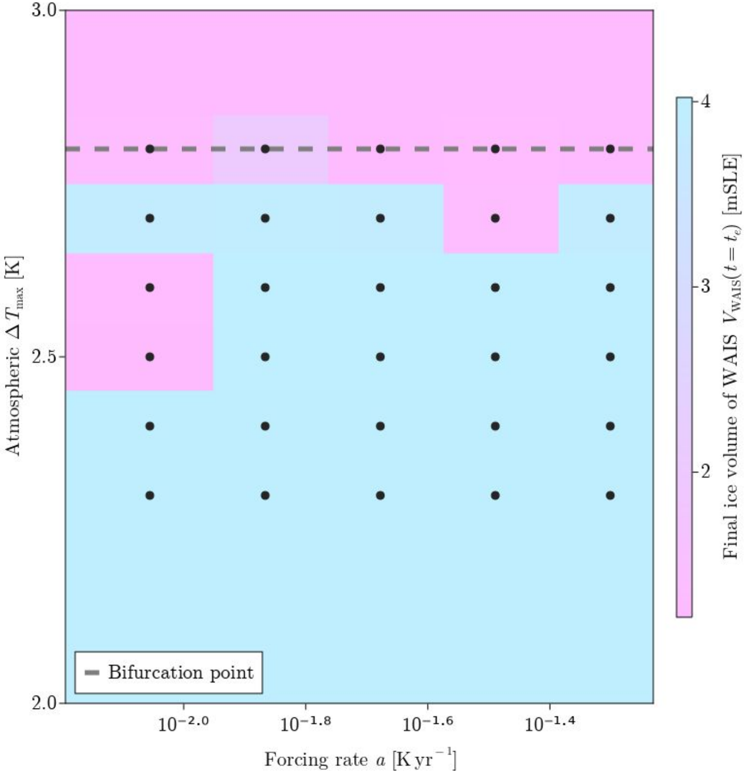
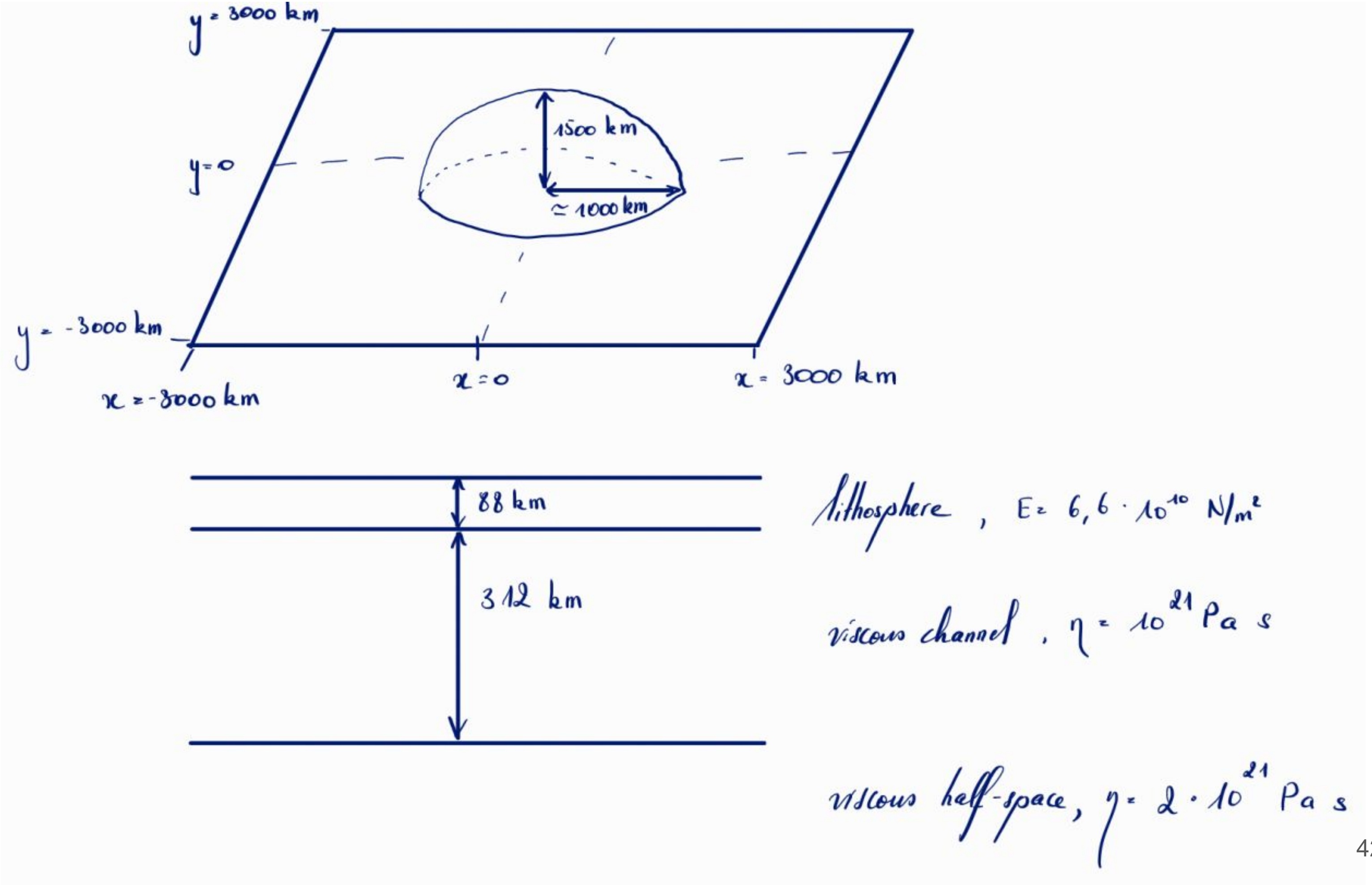



ELRA: τ = 1 yr



# Test 2



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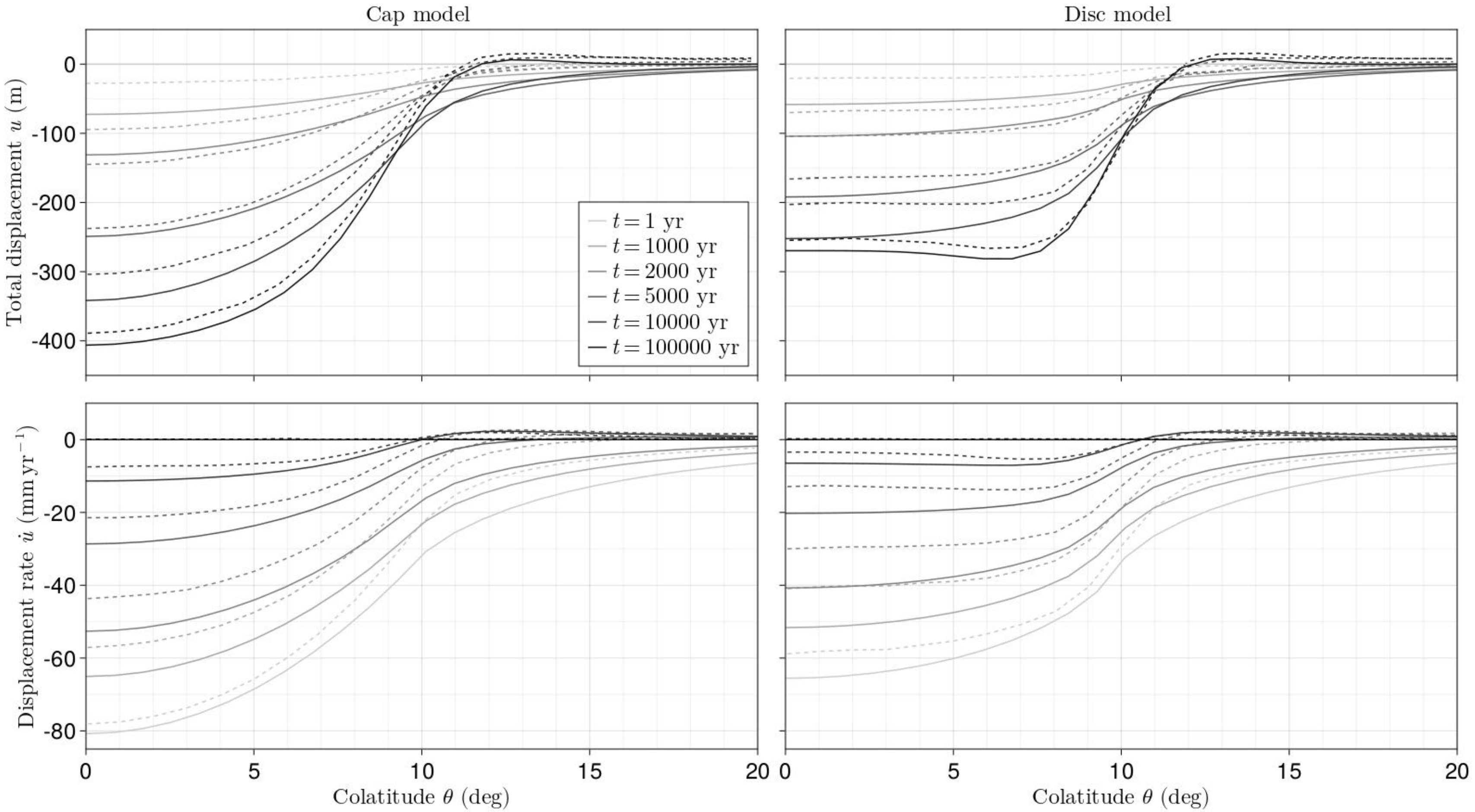
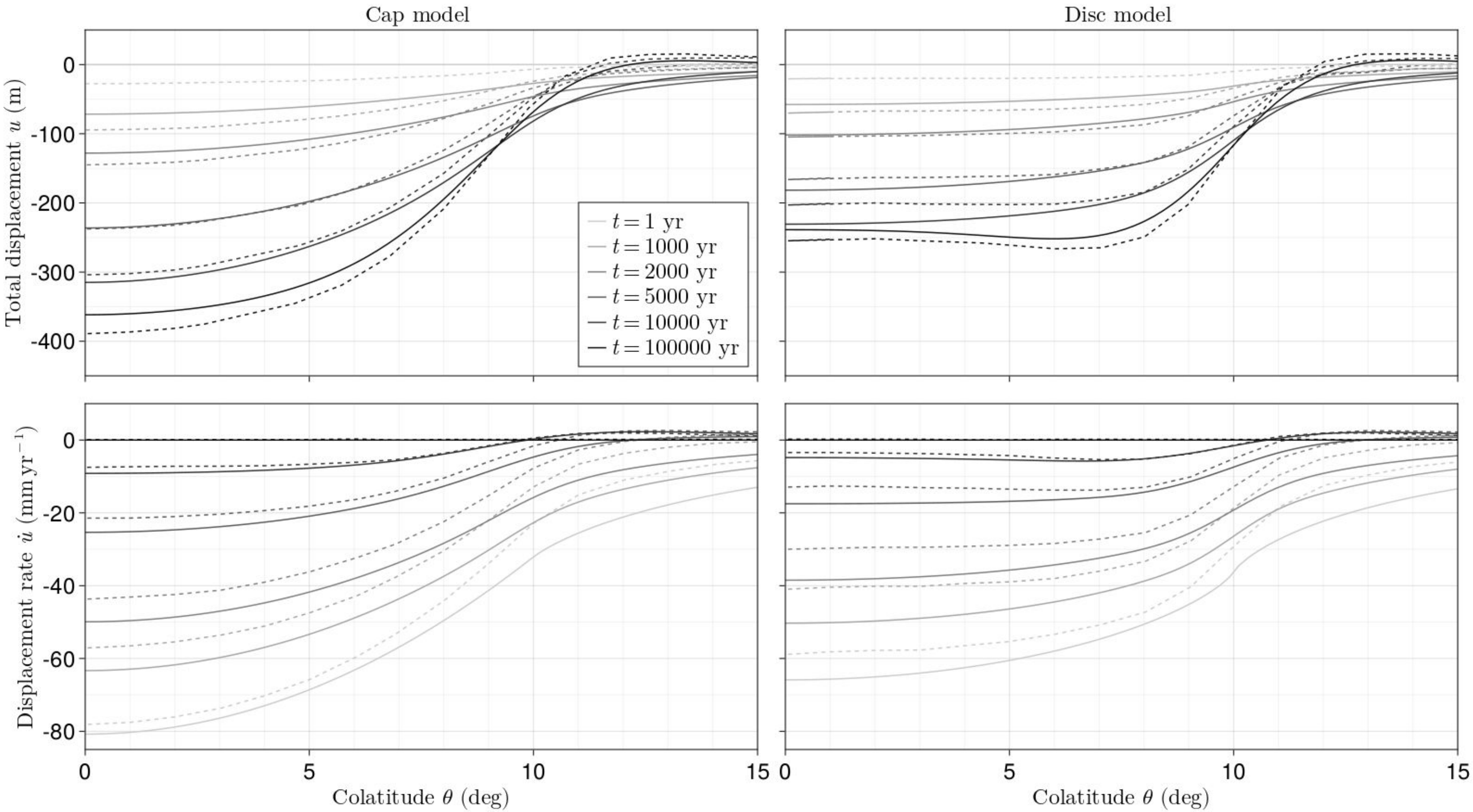
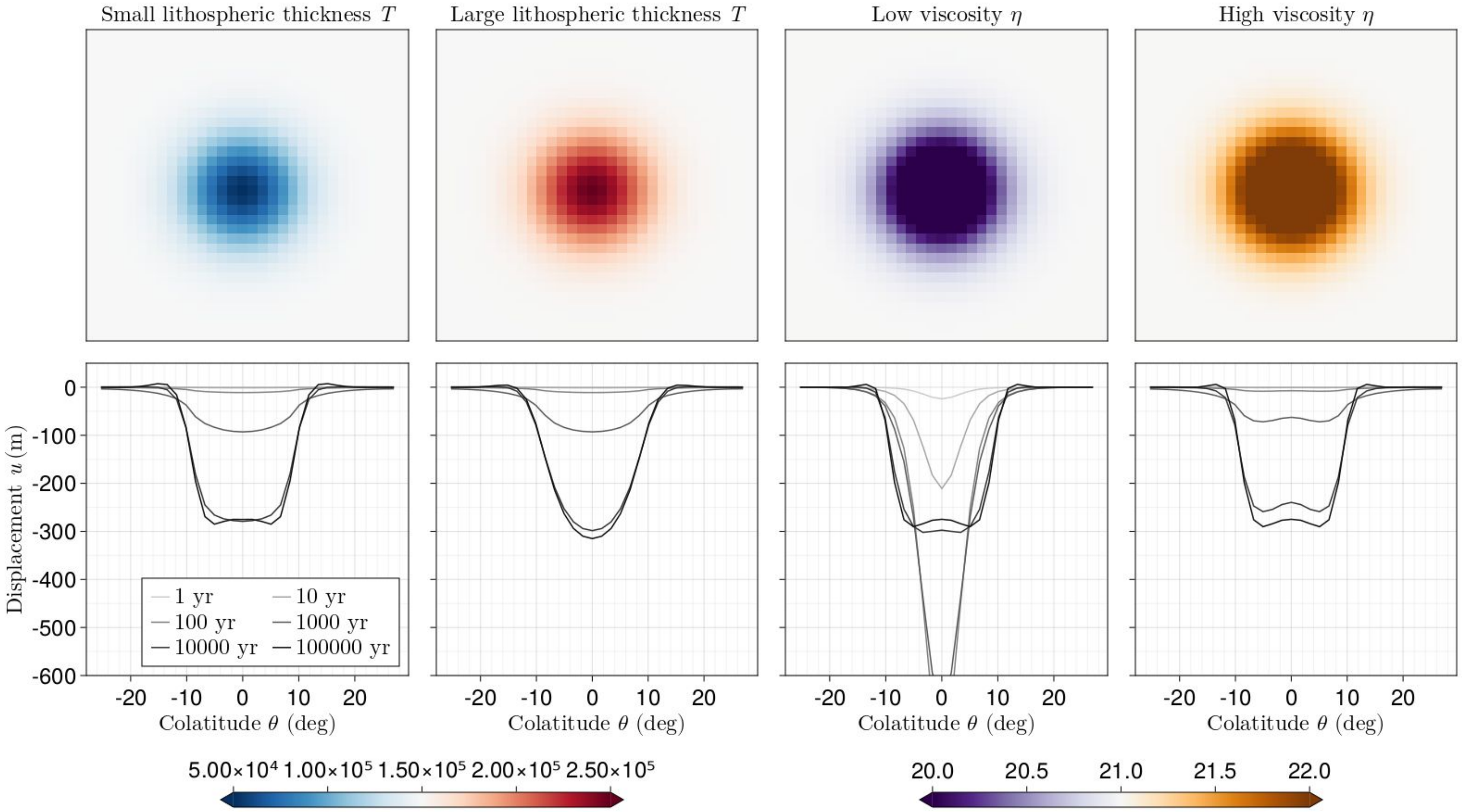
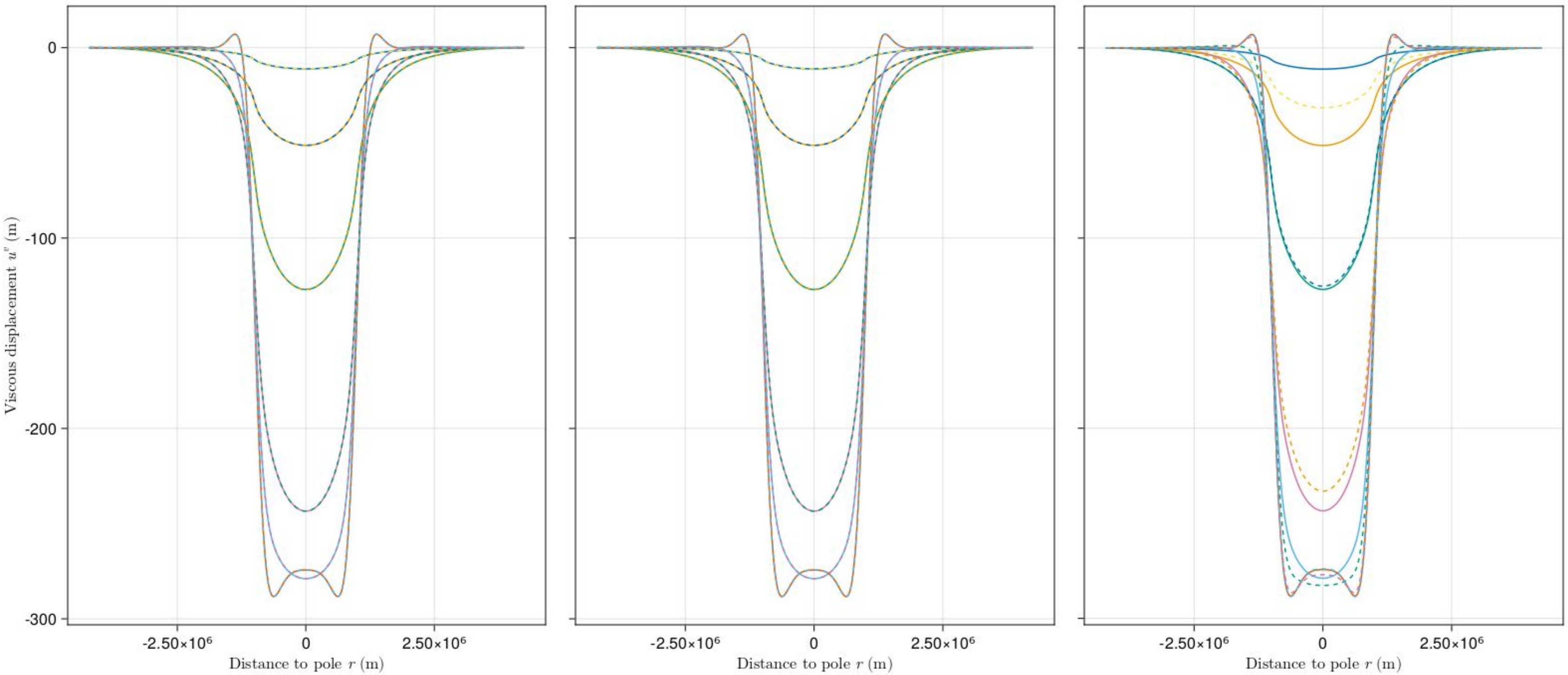


Cap model
Disc model
Total displacement $u$ (m)
Displacement rate $\dot{u}$ (mm yr$^{-1}$)
$t = 1$ yr
$t = 1000$ yr
$t = 2000$ yr
$t = 5000$ yr
$t = 10000$ yr
$t = 100000$ yr
Colatitude $\theta$ (deg)


Cap model
Disc model
Total displacement $u$ (m)
Displacement rate $\dot{u}$ (mm yr$^{-1}$)
$t = 1$ yr
$t = 1000$ yr
$t = 2000$ yr
$t = 5000$ yr
$t = 10000$ yr
$t = 100000$ yr
Colatitude $\theta$ (deg)
Colatitude $\theta$ (deg)
0
-100
-200
-300
-400
-20
-40
-60
-80
0
5
10
15


Small lithospheric thickness $T$
Large lithospheric thickness $T$
Low viscosity $\eta$
High viscosity $\eta$
Displacement $u$ (m)
0
-100
-200
-300
-400
-500
-600
1 yr
10 yr
100 yr
1000 yr
10000 yr
100000 yr
-20
-10
0
10
20
Colatitude $\theta$ (deg)
5.00×10⁴ 1.00×10⁵ 1.50×10⁵ 2.00×10⁵ 2.50×10⁵
20.0
20.5
21.0
21.5
22.0

# GPU vs. CPU, Euler vs. Crank-Nicolson, 2 vs. 3 layers



# Further modifications

- Boundary conditions: mean 0 displacement at corners of domain.
- GPU version of the code to make our speed argument even stronger.
- Allow $\eta = \eta(t)$ → makes simulation on long time scales easier than it has ever been!