A finite element based approach to computing balance velocities with application to Greenland mass balance

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19 June, 2013
Breckenridge, CO
We’ve become accustomed to thinking about extremes

*Extreme ice!!!*
Extremely boring: What if... the ice surface were stationary?
It follows that the thickness would be stationary.

from Bamber and Griggs 2012
Where would the mass go?

Surface Mass Balance Pattern
Problem Statement
A New Algorithm
Application
Results

What enters, must leave.
Flux balance

Change of thickness = mass in - mass out

The continuity equation

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\overline{u} H) + \dot{a}$$
What enters, must leave.

Flux balance

Change of thickness = mass in - mass out

The continuity equation

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\mathbf{u}H) + \dot{a}$$

Assume:

- Thickness, $H$ is known from radar measurements
- Accumulation, $\dot{a}$ is known from regional climate model.
- Thickness rate of change, $\frac{dH}{dt}$, is zero
- Vertically averaged velocity, $\mathbf{u}$ is directed downslope, e.g. $\mathbf{u} = |\mathbf{u}| \mathbf{N}$
- $\mathbf{N} \sim - (\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y})$
The balance velocity (speed)
Solution under the previously stated assumptions

The balance equation

\[ \nabla \cdot (|\mathbf{u}| \nabla SH) + \dot{a} = 0 \]

Solve for scalar speed, \(|\mathbf{u}|\).
Routing Algorithms

Solution under the previously stated assumptions

(a) Eight-direction Pour Point Model

(b) Grid of Terrain Elevations

(c) Grid of Flow Directions

(d) Drainage Network Showing Flowaccumulation
Routing Algorithms

Solution under the previously stated assumptions

- Each point depends on *all* those upslope from it.
- Must sort, and sequentially go from highest to lowest.
- Because of sort, algorithm is sequential.
- It is also hard to generalize to unstructured meshes.
Balance Speed

Enticing results
Balance Speed

Potential uses of better algorithm

- Model initialization; use of balance guarantees flux divergence equals surface mass balance
- Initial guess at basal traction, temperatures for balance velocities
- Basal water model treating water fluxes only
- Balance speeds for arbitrary ice configurations
- Actual speed could be computed if $\frac{\partial H}{\partial t}$ were known.
Smoothing the surface
Apply as PDE

Surface Smoothing PDE

\[ \tau_s = \nabla \cdot (lH)^2 \nabla \tau_s - \tau_d \]

and

\[ N = \frac{\tau_s}{\|\tau_s\|}. \]

- Direction of flow determined by solution of PDE,
- smooth the driving stress \( \tau_d \).
- \( l \) is adjusted as a smoothing length, here 12 is used.
- Eventually, more sophisticated models could be used.
The balance velocity (speed)
Solution under the previously stated assumptions

The balance equation, weak form, stabilized

\[ \int_{\Omega} \left( \lambda + \gamma \nabla \cdot \mathbf{N} \mathbf{H} \lambda \right) \left( \nabla \cdot \mathbf{N} \mathbf{H} \dot{\mathbf{U}} - \dot{\mathbf{a}} \right) \, d\Omega = 0 \]

Stabilization parameter

\[ \gamma = \frac{h}{2 \| \mathbf{N} \mathbf{H} \|}, \]

- \( \lambda \) is a test function, \( h \) the mesh size.
- The key is application of stabilization to flux, and not velocity.
- Solutions are now robust, independent of mesh, efficient, and scalable to parallel architectures. (800K DOF, 1.5 minutes, 4 cores)
Actual rates of dynamic thinning, $\frac{\partial H}{\partial t}$

Can now find “actual” velocity, (Csatho personal communication)
Imbalance Speed 2003-04

\[ \frac{\partial H}{\partial t} = - \nabla \cdot (|\mathbf{u}| \mathbf{N} H) + \dot{a} \]
Imbalance Speed 2003-04

\[ \frac{\partial H}{\partial t} = -\nabla \cdot (|\mathbf{u}| \mathbf{N} H) + \dot{a} \]

- Take \( \frac{\partial H}{\partial t} \) from altimeter data, removing firn and SMB signals.
- Use modeled SMB (RACMO).
- Use \( H \) from recent radar surveys.
- Smoothed directions, \( \mathbf{N} \).
Imbalance Speed 2004-05
Imbalance vs. Observed Speed 2005-06

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balance velocity 0506

U obs 2005 2006

Imbalance vs. Observed Speed 2005-06
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Imbalance vs. Observed Speed 2006-07
Imbalance vs. Observed Speed 2007-08
Speed differences from balance 2005-06
How “out of balance” is the velocity?

Computed difference from balance

Observed difference from balance
Speed differences from balance 2006-07
How “out of balance” is the speed?

Computed difference from balance

Observed difference from balance
Speed differences from balance 2007-08
How “out of balance” is the speed?

Computed difference from balance  

Observed difference from balance
Conclusions

- A new, efficient algorithm makes balance speeds easy to compute.
- The algorithm may see application in a number of areas; initialization, state estimation, basal water routing, etc.
- Combined with surface mass balance and thickness rate of change data, actual velocities can be computed.
- It is likely that the errors in these calculations provide are larger than those found from InSAR, but, the coverage is better.
- Errors in thickness, smb, and dhdt likely to decrease in coming years.
- N can be replaced with other directions, such as those coming from a HO model.