Preconditioning Techniques Based on Domain Decomposition Methods

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June 21, 2012

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Introduction

Domain decomposition methods:
the process of subdividing the solution of a large system into smaller subproblems whose solutions can be used to produce a preconditioner for the system of equations that results from discretizing the PDE on the entire domain

Idea of Domain Decomposition Methods

- Decompose the domain Ω into overlapping or non-overlapping subdomains.
- Assign one or several subdomains to each processor of parallel machine.

In each iteration:

- In each subdomain, solve small local subproblems.
- In addition, solve one small global problem.

Motivation

Conventional methods

- we usually need additional information, e.g., coarse coordinate information.
- we need quite regular meshes.
- it is hard to apply for irregular subdomains.

Alternative Approach

Generalized Dryja, Smith, Widlund (GDSW) coarse space technique

- this technique is based on energy minimizing discrete harmonic extensions.
- it has been applied to many applications
    - almost incompressible elasticity (Dohrmann, Widlund)
    - Reissner-Mindlin plates (Lee)
    - Raviart-Thomas vector fields (Oh)

Alternative Approach

Advantage

- the method can be implemented in an algebraic manner - we do not need any coarse discretization.
- it works well for irregular subdomains and unstructured meshes.
- it has well-established theoretical results, e.g., upper bounds of condition number.

Discrete Harmonic Extension

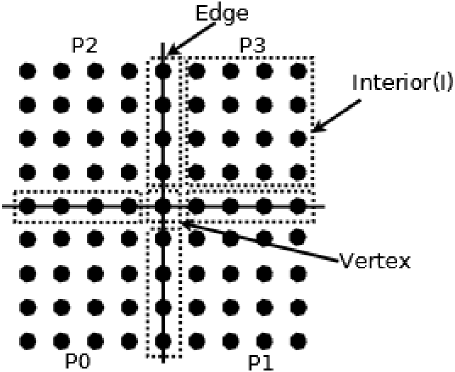
A vector $u^{(i)} := [u_I^{(i)T} u_\Gamma^{(i)T}]^T$ is said to be discrete harmonic on Ω_i if

$$A_{II}^{(i)} u_I^{(i)} + A_{I\Gamma}^{(i)} u_\Gamma^{(i)} = 0.$$

$u^{(i)}$ is completely defined by $u_\Gamma^{(i)}$.
The discrete harmonic extension has the minimal energy property.

$$\mathbf{a}(\mathbf{u}, \mathbf{u}) = \min_{\mathbf{v}|_\Gamma = \mathbf{u}_\Gamma} \mathbf{a}(\mathbf{v}, \mathbf{v})$$

Coarse Component



Interface(Γ) : Vertex + Edge

Coarse Component

- R_0 : restriction to coarse space
  - We choose one coarse edge or vertex and give 1 to the nodes on the edge or vertex.
  - We assign 0 to other nodes on the interface.
  - We use the discrete harmonic extension for interior parts.
- A_0 : $R_0 A R_0^T$

We note that this coarse component can be implemented in an algebraic manner. We do not need any coarse discretizations.

Additive Schwarz Perconditioner

Additive Schwarz Method for SPD systems

$$P^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$$

- A_0 : coarse matrix (restriction to the coarse space)
- A_i : local matrix (restriction to overlapping subdomain Ω_i')
- R_0 : restriction to coarse space
- R_i : restriction to overlapping subdomain Ω_i'

Restricted Additive Schwarz Perconditioner

Restricted Additive Schwarz Method for indefinite or nonsymmetric systems

$$P^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} \widetilde{R}_i^T A_i^{-1} R_i$$

- A_0 : coarse matrix (restriction to the coarse space)
- A_i : local matrix (restriction to extended subdomain Ω_i')
- R_0 : restriction to coarse space
- R_i : restriction to overlapping subdomain Ω_i'
- $\widetilde{R}_i$: restriction to subdomain Ω_i

Numerical Experiments

5km Greenland Ice-Sheet
1 subdomain per each processor, preconditioned GMRES
local solver : Amesos KLU
coarse solver : Amesos KLU

Table: iteration counts

# of processors	64	128	256	512
Ifpack ILU	227	269	310	307
DD	17	20	21	29