A Patch Recovery Interpolation method

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Interpolating atmospheric winds

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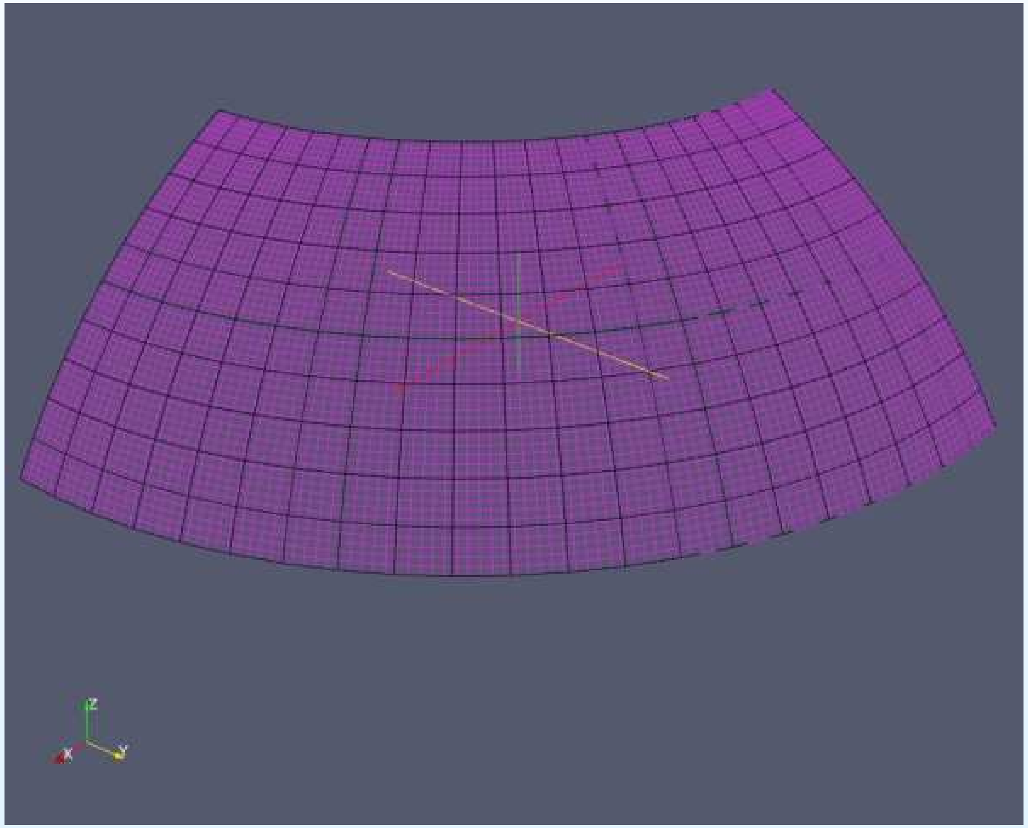

Example:Interpolating atmospheric winds, ...

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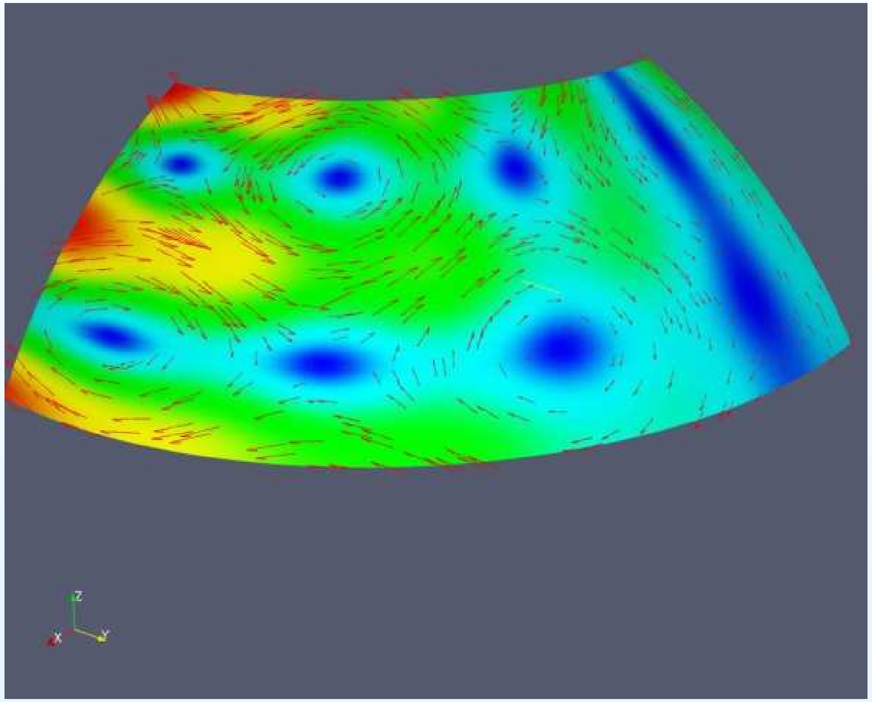

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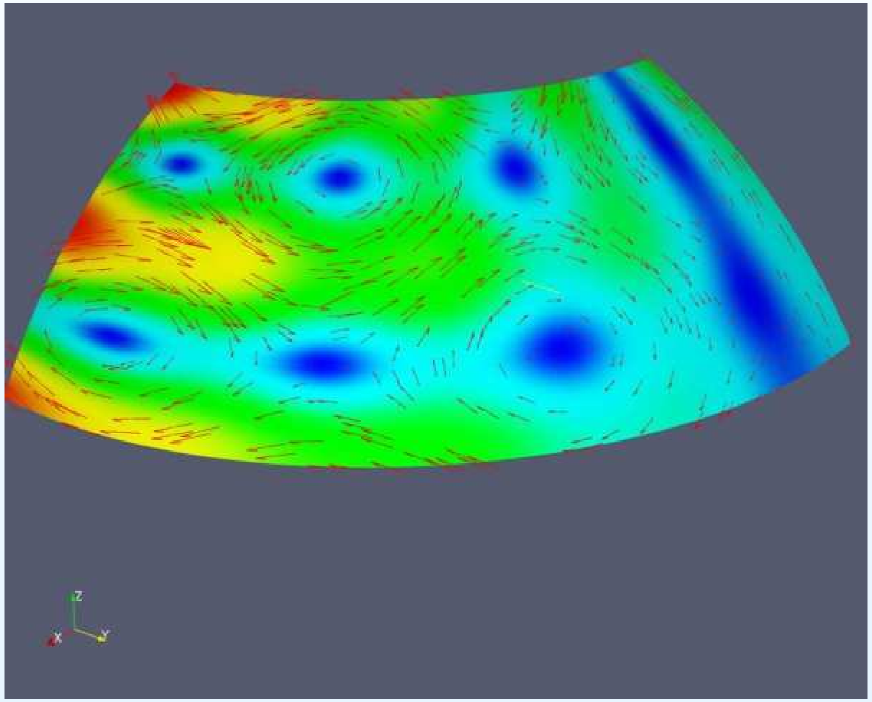

Of interest is the surface stress $\tau = C_p|U|U$, where $U = (u, v)$, especially $\nabla \times \tau$.

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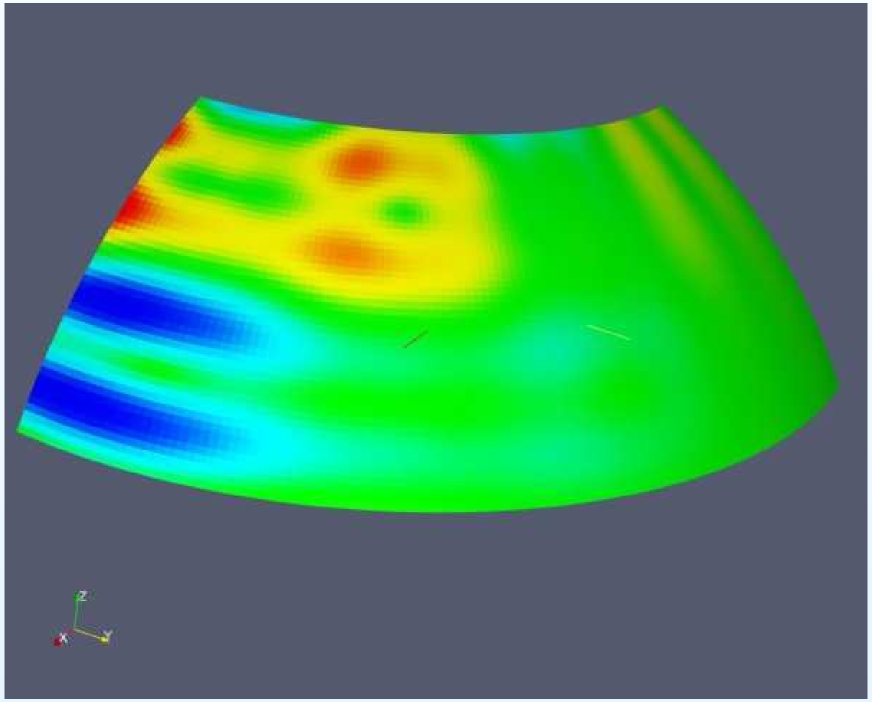

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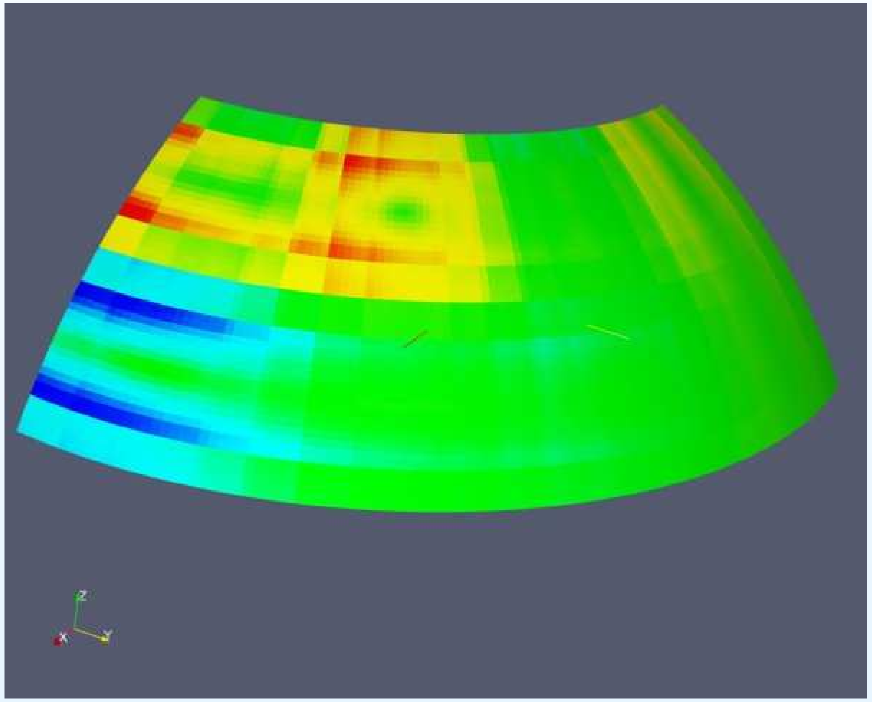

Curl of the standard bi-linear interpolant is not!

Computation aspects of interpolation

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Each of these topics is its own talk. We begin with the Interpolation method.

Bi-linear interpolation

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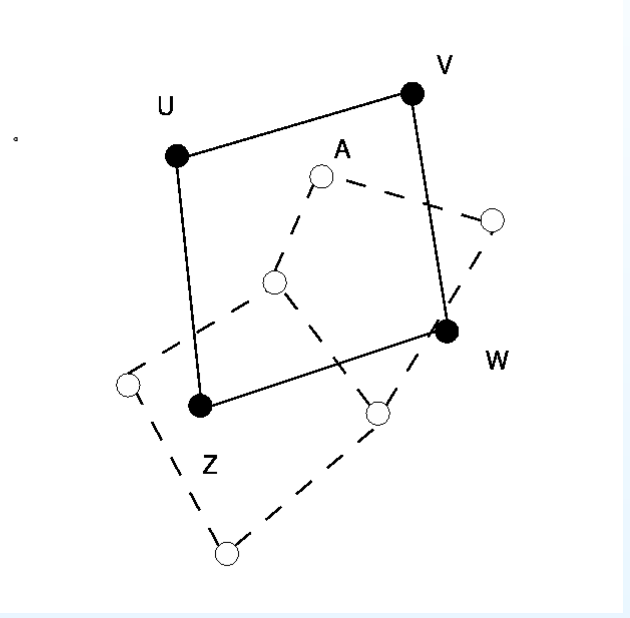

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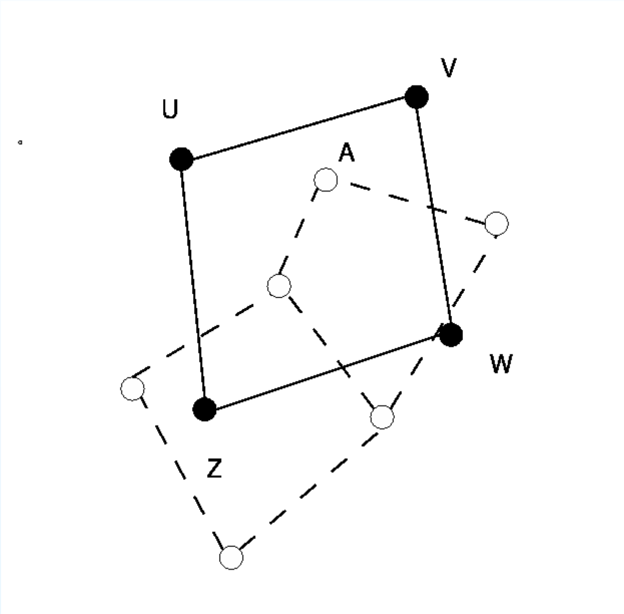

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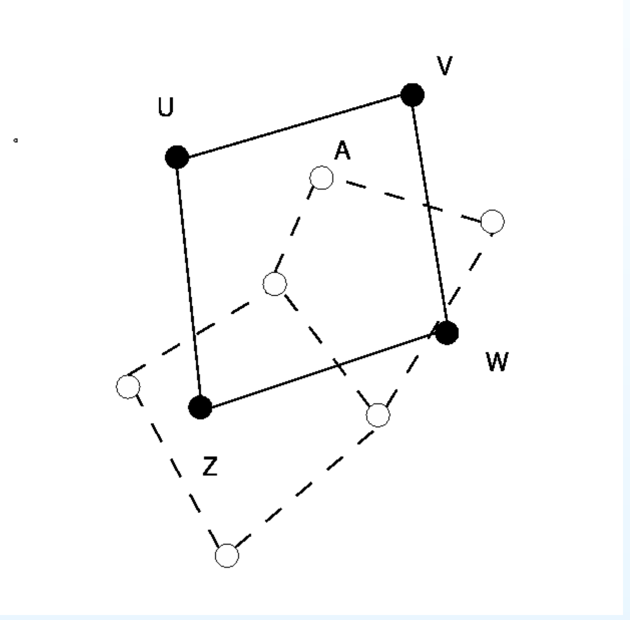

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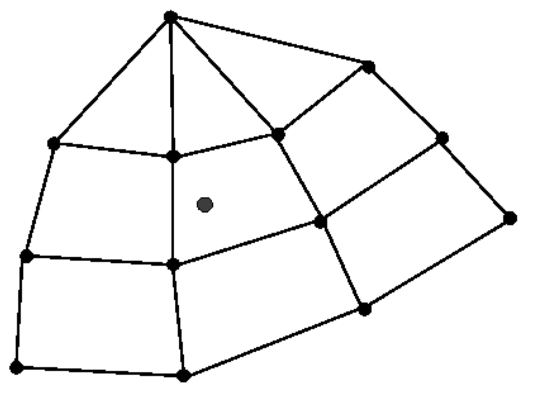
A reasonable approximation to ∇A is $U\nabla\phi_1 + \cdots + Z\nabla\phi_Z$.

Patch based methods

We form the interpolant at • using polynomials based on the node patches of the encompassing cell:



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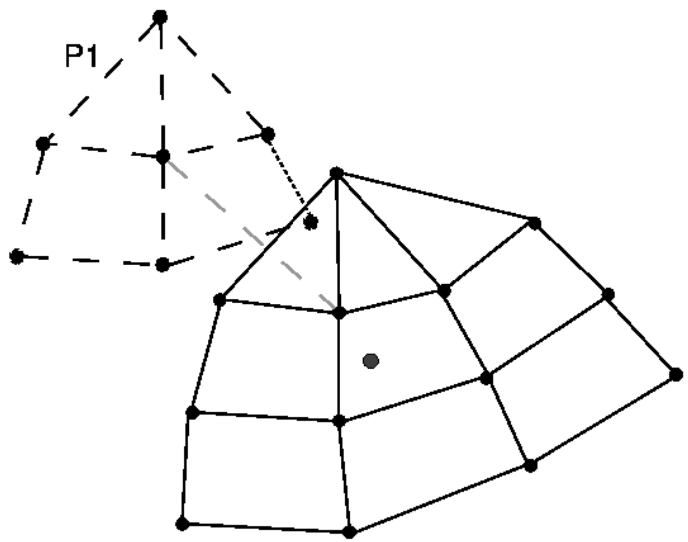

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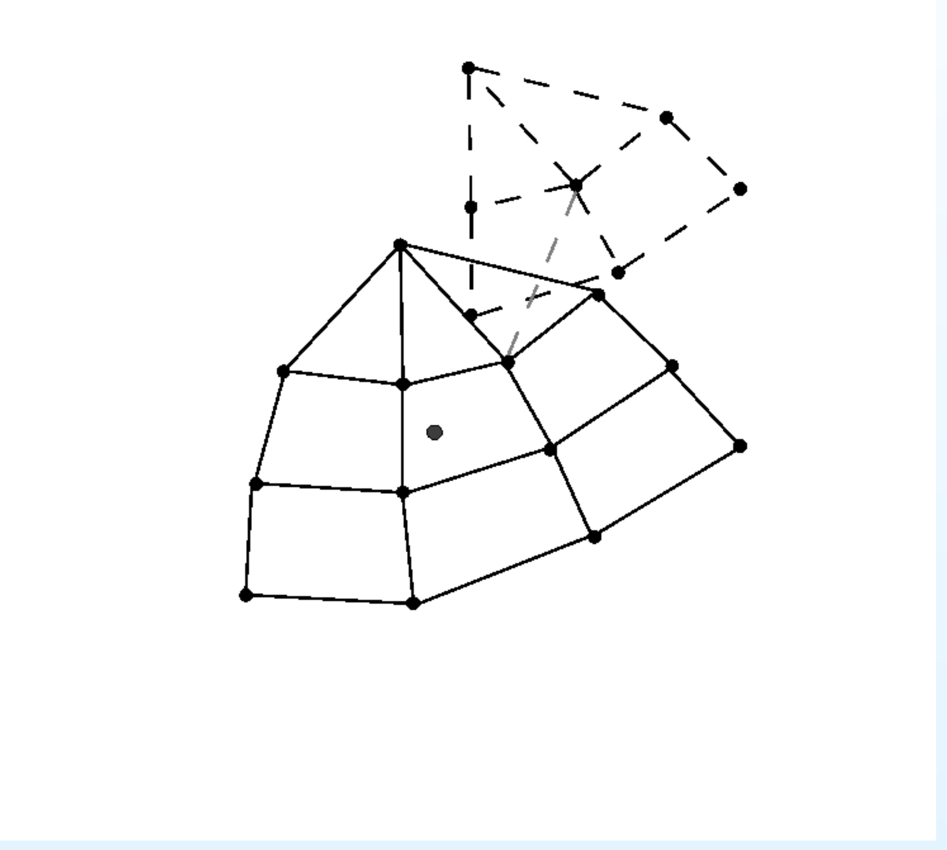

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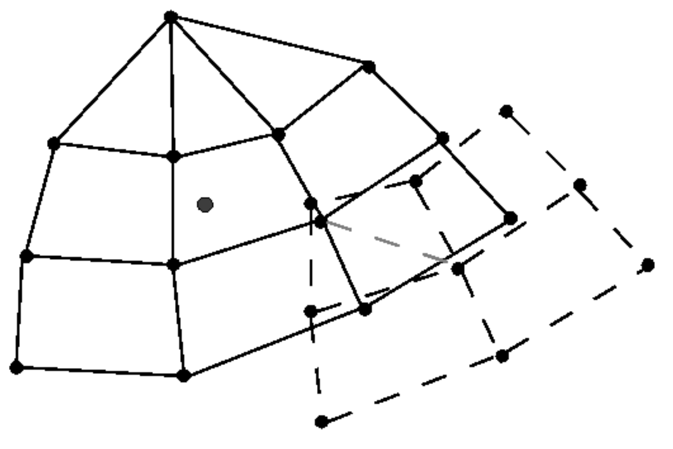

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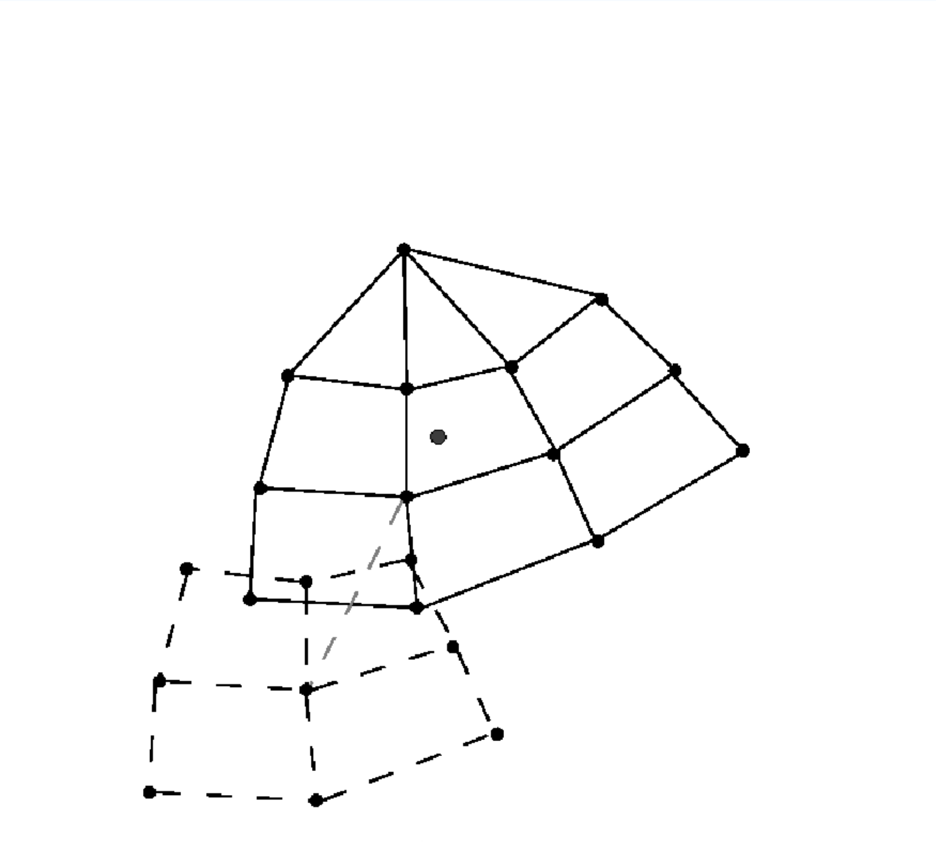
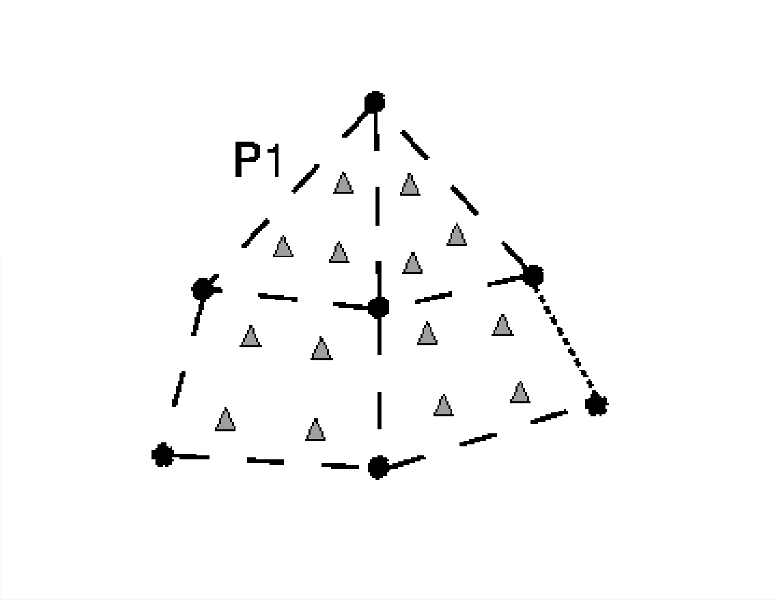


Patch based methods,...

On each patch we sample the source function at a set of sample points (usually quadrature points) $\triangle$, using local bi-linear interpolation if necessary.



Call these samples s_i at (local 2D) coordinates p_i.

Local polynomial approximation

We fit a tensor product polynomial through these values, solving for the polynomial coefficients c

$$\min_c \sum_i (Q(c, p_i) - s_i)^2$$

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On a manifold, the local coordinates p_i may either be full 3D coordinates, or the coefficients of the co-space of a reasonable normal.
This avoids pole type singularities in the patch algorithm (i.e. don't use lat/lon).

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We use any partition of unity on the cell to blend the patches for a value $F(x) = \sum_j \psi_j(x) Q(x)$, for instance the bi-linear basis.

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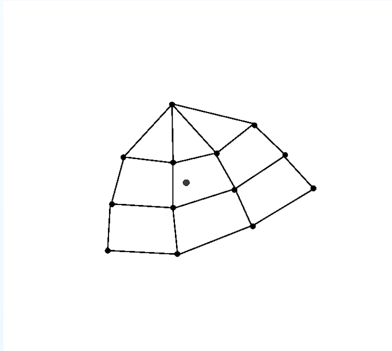

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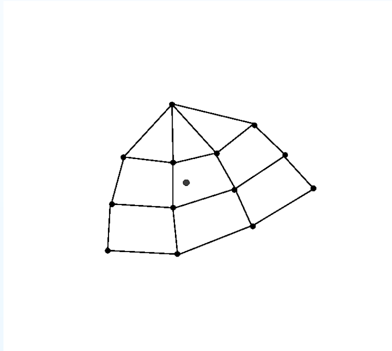

Explicitly, accounting for the local coordinate system $p = L(x)$ and the bi-linear interpolation to sample locations $s = \Phi f$, the interpolant is a linear function of the coefficients f on this enlarged stencil,

$$F(x) = \sum_j \left[\psi(x) \big(b \circ L(x) \big)^\top (A^\top A)^{-1} \Phi \right]_j f$$

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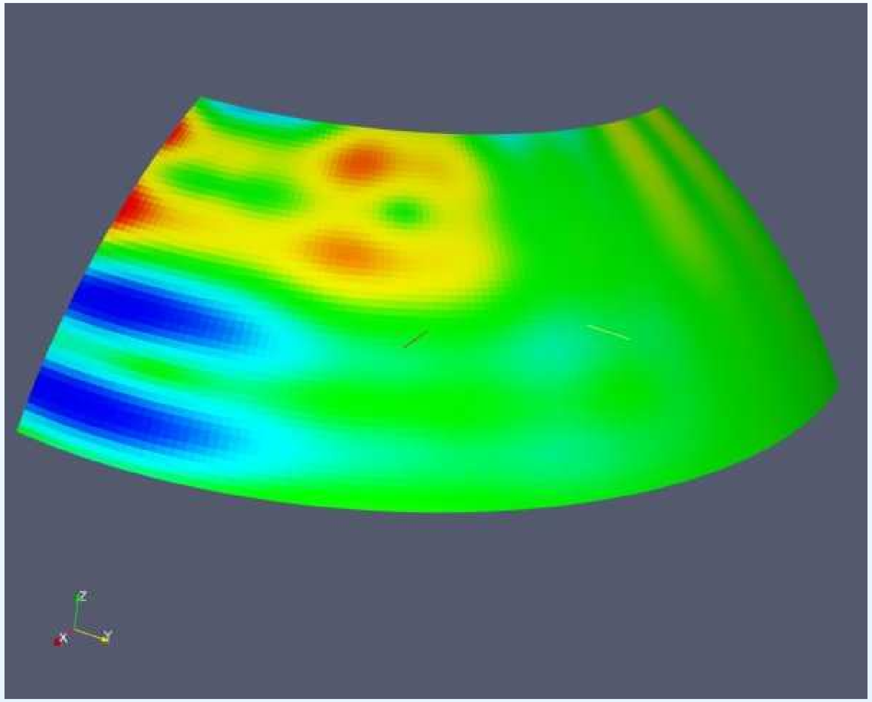

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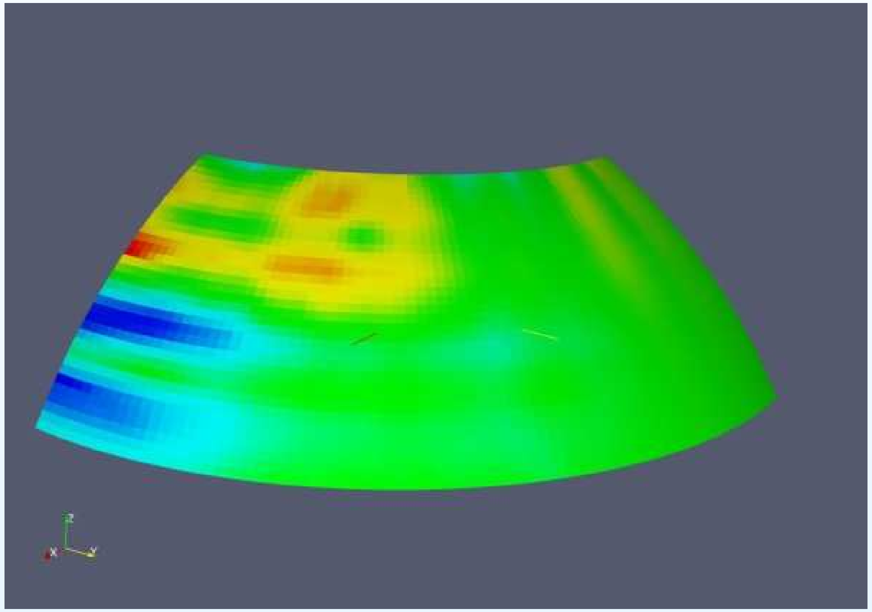

The patch recovery curl is far more reasonable compared to the bi-linear!

Some results from interpolation theory

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Smoothness is required, at least of weak derivatives $||D^2f||_{L^2}$.

An experiment

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$$f(x, y) = (1 - xy) \sin 3\pi x \cos 2\pi y)$$

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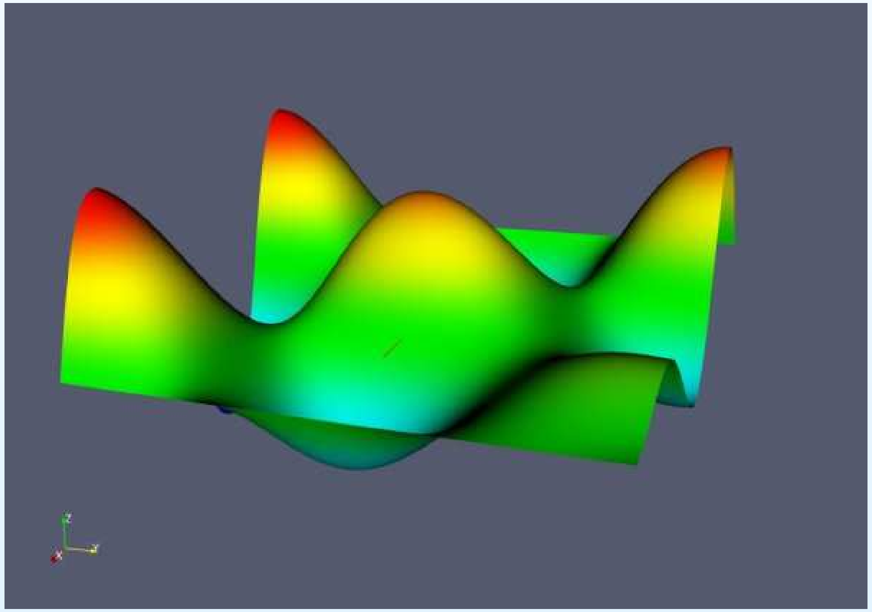

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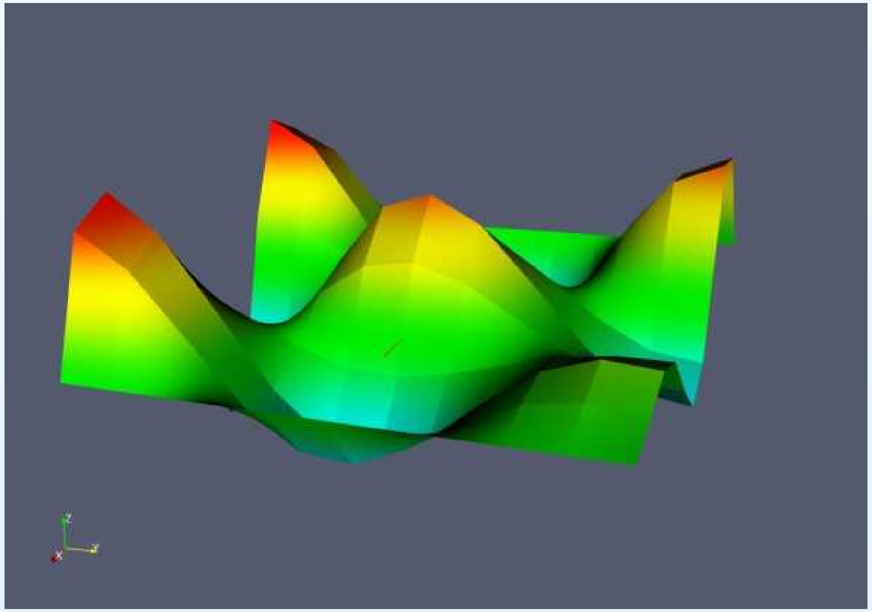
Bilinear



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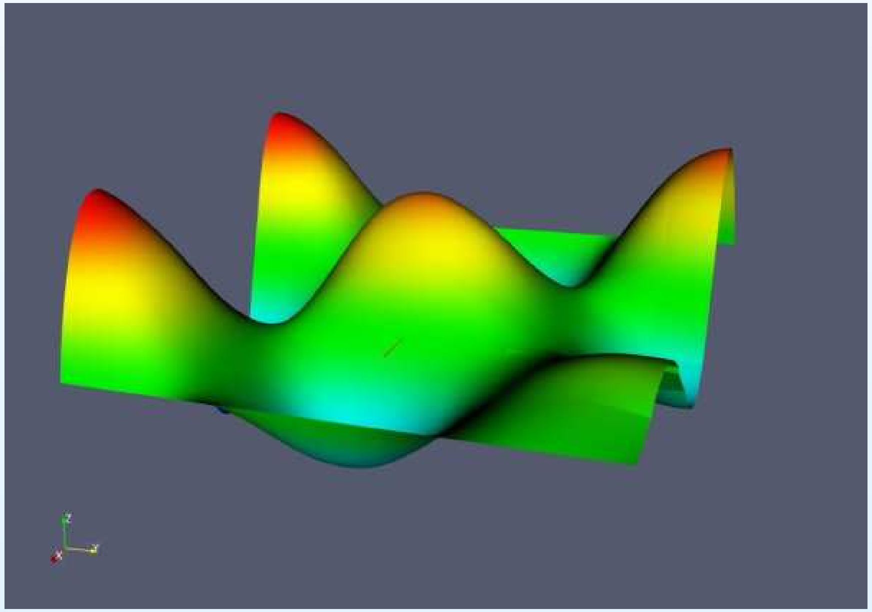

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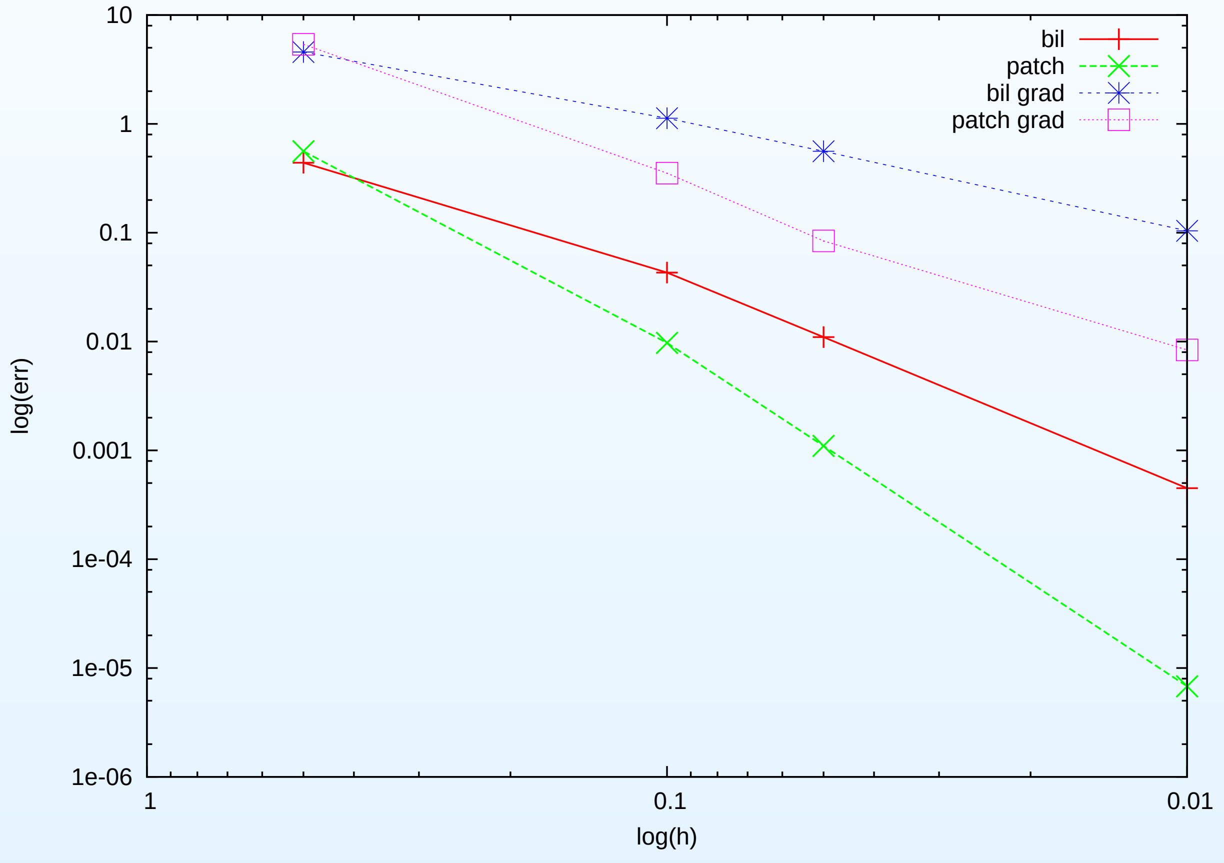

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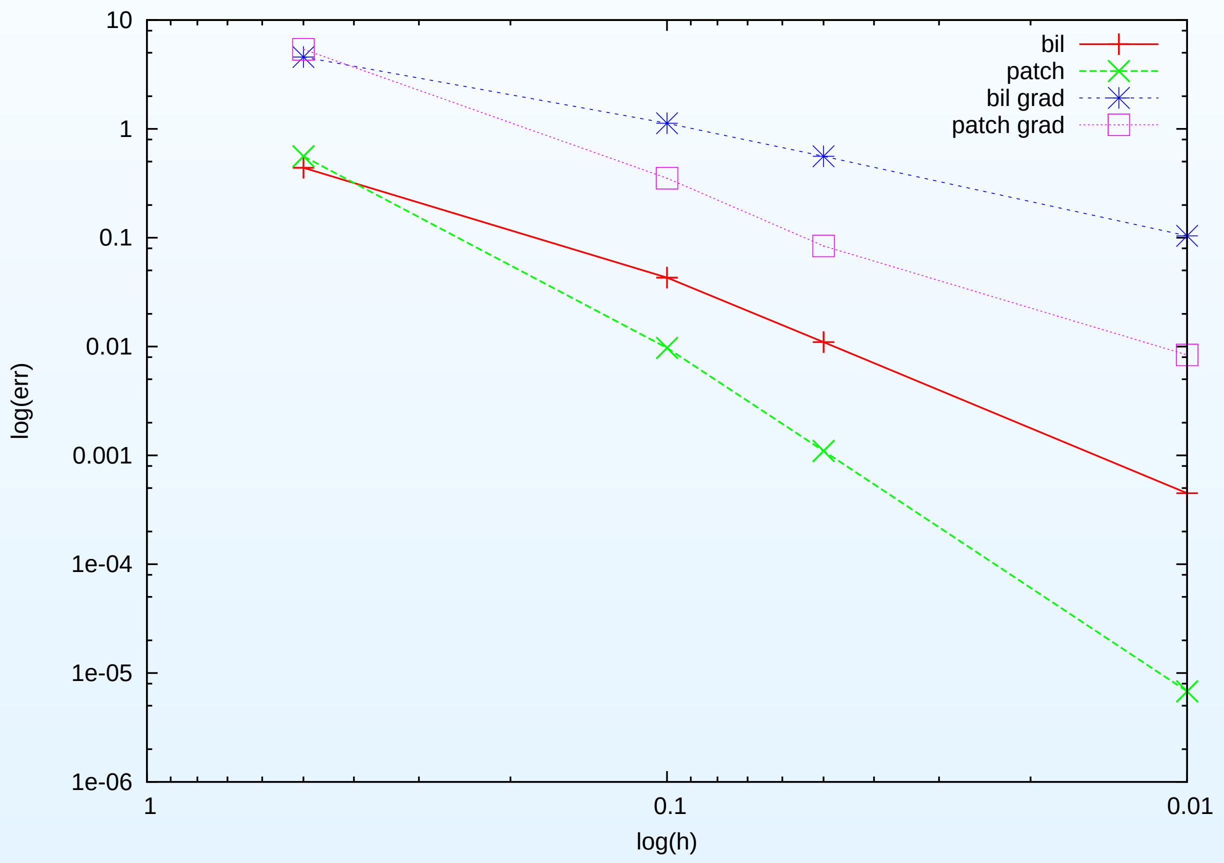

Rates are $P = 3.14, B = 1.96, \nabla P = 2.01, \nabla B = 1.01$.

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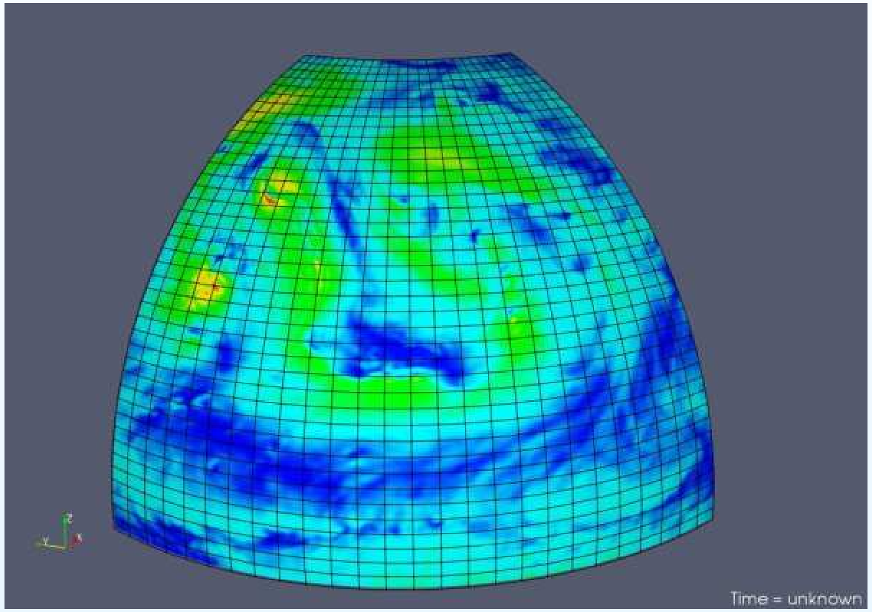

The exact wind field ($|U|$)

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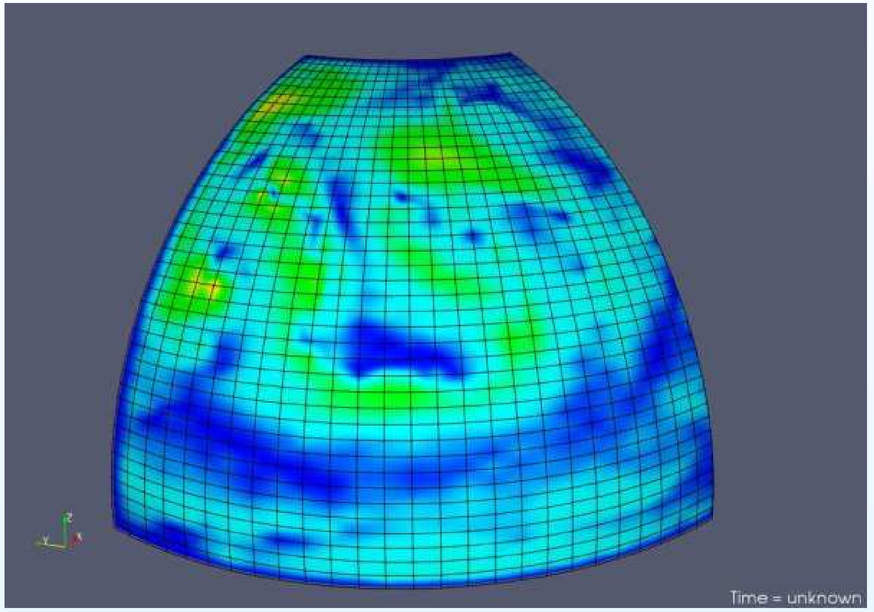

The bi-linear interpolant

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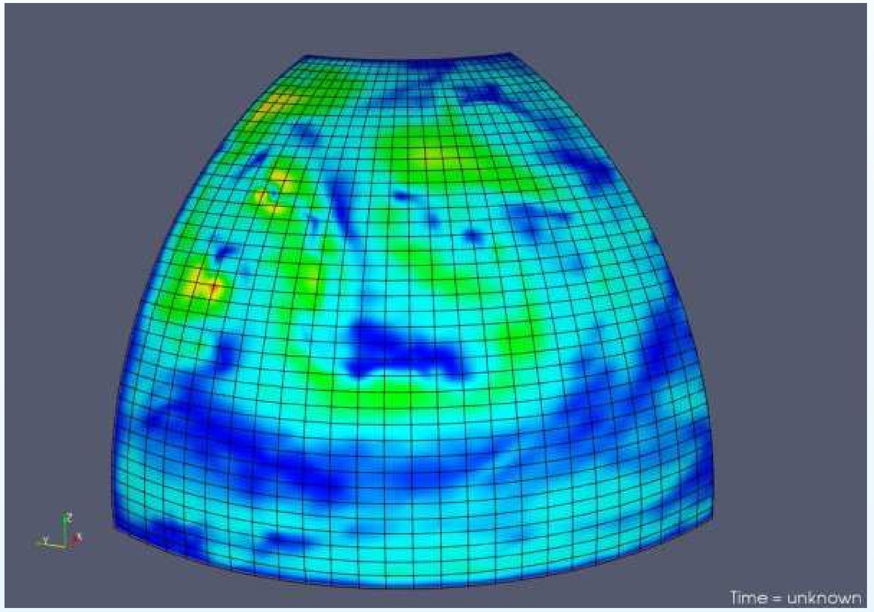

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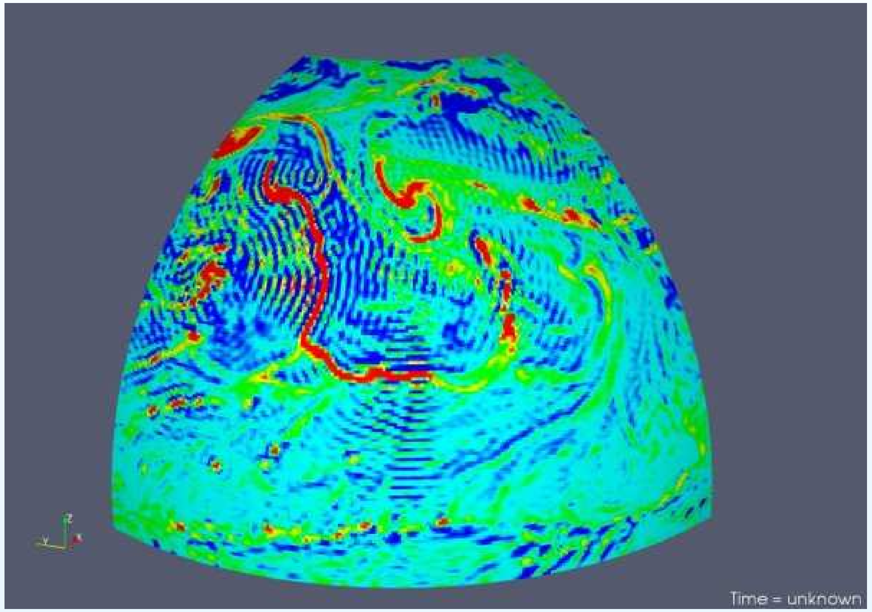

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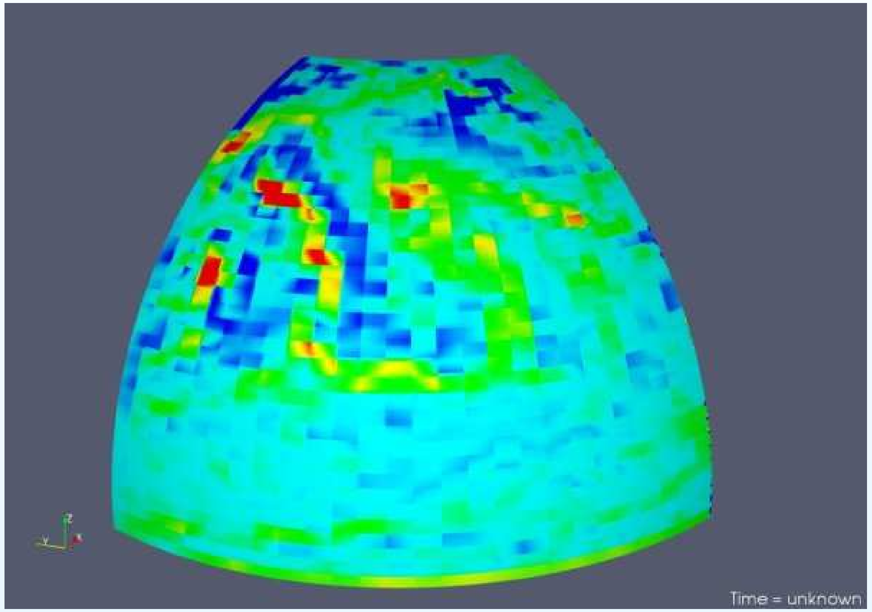

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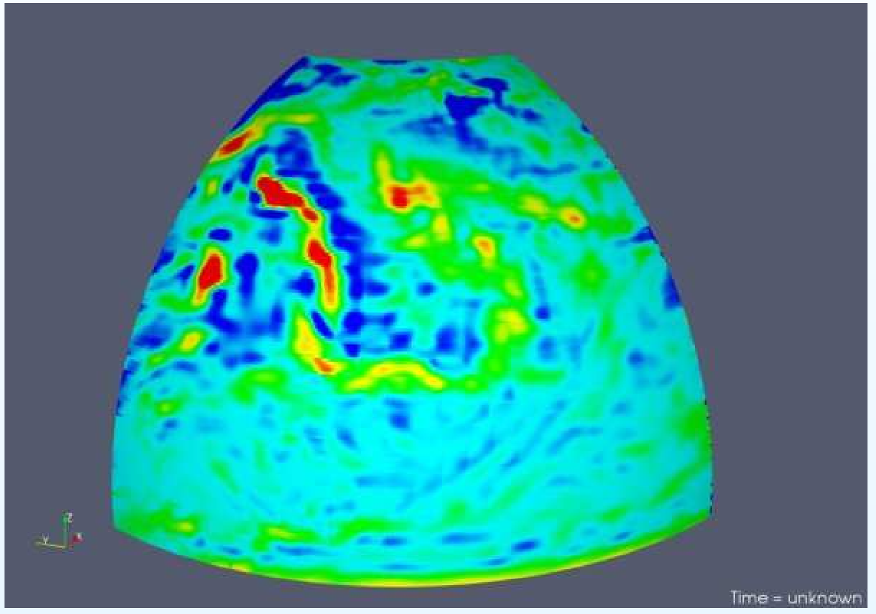

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The End