Application of physics-based interpolation to cryospheric data

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Geophysical data with noise and gaps in coverage

Problem Statement

Radar outstanding, but only along flightlines.
Geophysical data with noise and gaps in coverage

Problem Statement

InSAR velocity excellent, but has gaps and noise.
Geophysical data with noise and gaps in coverage

**Problem Statement**

Interpolation of bed produces artefacts.

*Johnson and Brinkerhoff*

*Physics-based interpolation*
Geophysical data with noise and gaps in coverage

Problem Statement

Close inspection of speed reveals noise.
Problem Statement

Data are combined to produce flux divergence, \( \nabla \cdot (\mathbf{uH}) \neq \dot{a} \)

Solution Strategy

Results

Geophysical data with noise and gaps in coverage

Problem Statement
Geophysical data with noise and gaps in coverage

Problem Statement

Prognostic modeling

- Assimilation of surface velocity (traction control variable)
- Steady state temperature
- Prognostic run forward
Objective
What do we hope to accomplish?

We seek to *reduce noise and interpolate* geophysical data in a manner is consistent with:

- other observations.
- physics.
- stated errors.
- smoothness requirements.

We like to call this *physics based interpolation*. The transient portion of prognostic runs should be removed, or greatly reduced.
Optimization cartoon

*With (slightly dated) references to popular culture!*
Optimization cartoon

*With (slightly dated) references to popular culture!*

**Objective Function**

\[ I = \int_{\Omega} \frac{1}{2} (u_m - u_o)^2 \, dx + \lambda \int_{\Omega} \left( \nabla \cdot u_m \hat{N} H - \dot{a} \right) \, dx \]
Optimization cartoon

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**Minimum**

(low point)

**Constraint (road)**
Optimization cartoon

The bounds come from data

Objective Function
\[ I = \int_{\Omega} \frac{1}{2} (u_m - u_o)^2 \, dx + \lambda \int_{\Omega} \left( \nabla \cdot u_m \hat{N} H - \hat{a} \right) \, dx \]

Uncertainty in Observation, e.g.
\[ H \in [H_o - \Delta H_o, H_o + \Delta H_o] \]

Minimum (low point)
Constraint (road)

Road width

Johnson and Brinkerhoff
Physics-based interpolation
A closer look at error bounds
Error bounds enter the constraint

$$H \in [H_0 - \Delta H_0, H_0 + \Delta H_0]$$

$$u_0 \in [u_0 - \Delta u_0, u_0 + \Delta u_0]$$

$$\dot{a} \in [\dot{a} - \Delta \dot{a}, \dot{a} + \Delta \dot{a}]$$

$$\hat{N} \in [\hat{N} - \Delta \hat{N}, \hat{N} + \Delta \hat{N}]$$

Speed errors published with InSAR data, note $u_0 < .5$ discarded
A closer look at error bounds
Error bounds enter the constraint

\[ H \in [H_o - \Delta H_o, H_o + \Delta H_o] \]
\[ u_o \in [u_o - \Delta u_o, u_o + \Delta u_o] \]
\[ \dot{a} \in [\dot{a} - \Delta \dot{a}, \dot{a} + \Delta \dot{a}] \]
\[ \hat{N} \in [\hat{N} - \Delta \hat{N}, \hat{a} + \Delta \hat{N}] \]

Thickness errors published with Bamber 2013 bed topography, note min. error of 35 m imposed.
A closer look at error bounds

Error bounds enter the constraint

Error bounds

\[ H \in [H_0 - \Delta H_0, H_0 + \Delta H_0] \]
\[ u_0 \in [u_0 - \Delta u_0, u_0 + \Delta u_0] \]
\[ \dot{a} \in [\dot{a} - \Delta \dot{a}, \dot{a} + \Delta \dot{a}] \]
\[ \hat{N} \in [\hat{N} - \Delta \hat{N}, \hat{a} + \Delta \hat{N}] \]

No idea of the errors in apparent mass balance. Guess ± 10 m.
A closer look at error bounds

Error bounds enter the constraint

\begin{align*}
H & \in [H_0 - \Delta H_0, H_0 + \Delta H_0] \\
u_0 & \in [u_0 - \Delta u_0, u_0 + \Delta u_0] \\
\dot{a} & \in [\dot{a} - \Delta \dot{a}, \dot{a} + \Delta \dot{a}] \\
\hat{N} & \in [\hat{N} - \Delta \hat{N}, \hat{N} + \Delta \hat{N}] 
\end{align*}

Errors in $\hat{N}$ estimated to be $\pm 5^\circ$ for fast moving ice and $\pm 1^\circ$ elsewhere.
Optimization cartoon

Inside the BFGS, destination is needed

Where are we going?
Optimization cartoon
The destination is the *data*
Optimization cartoon

Current location is also needed, this is the model output

Where are we?
Optimization cartoon
Current location is also needed, this is the model output

Johnson and Brinkerhoff
Physics-based interpolation
Optimization cartoon
The directions are challenging to understand

How do we get there?
Optimization cartoon
The directions are challenging to understand

\[ \delta I = \int_{\Omega} \left[ (u_m - u_o) \delta u_m - (u_m - u_o) \delta u_o \right] dx \]
\[ + \chi \int_{\Omega} \left[ \nabla \cdot (\delta u_m \hat{N} H) + \nabla \cdot (u_m \tilde{N} \delta H) + \nabla \cdot (u_m H \delta \hat{N}) - \delta \hat{a} \right] dx \]
\[ + \delta \lambda' \int_{\Gamma} \left( \nabla \cdot u_m \hat{N} H - \hat{a} \right) dx \]
Optimization cartoon
The directions are challenging to understand
Gradients explained
Optimization requires *gradients*

Chain rule variation of objective function

\[
\delta I = \delta I(\delta H, u_0, \dot{a}, \hat{N}, \lambda') + \delta I(H, \delta u_0, \dot{a}, \hat{N}, \lambda') \\
+ \delta I(\delta H, u_0, \delta \dot{a}, \hat{N}, \lambda') + \delta I(\delta H, u_0, \dot{a}, \delta \hat{N}, \lambda') \\
+ \delta I(\delta H, u_0, \dot{a}, \hat{N}, \delta \lambda')
\]
Gradients explained
Optimization requires *gradients*

Chain rule variation of objective function

\[
\delta I = \delta I(\delta H, u_o, \dot{a}, \hat{N}, \lambda') + \delta I(H, \delta u_o, \dot{a}, \hat{N}, \lambda') \\
+ \delta I(\delta H, u_o, \delta \dot{a}, \hat{N}, \lambda') + \delta I(\delta H, u_o, \dot{a}, \delta \hat{N}, \lambda') \\
+ \delta I(\delta H, u_o, \dot{a}, \hat{N}, \delta \lambda')
\]

Find a variation, for example, \(\delta H\)

\[
\delta I(\delta H, u_o, \dot{a}, \hat{N}, \lambda) = \int_{\Omega} \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} I(H + \epsilon \delta H, u_o, \dot{a}, \hat{N}, \lambda) \, dx,
\]

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Physics-based interpolation
Gradients explained
Optimization requires *gradients*

Application of variation throughout

\[
\delta I = \int_{\Omega_e} \left[ (u_m - u_o) \delta u_m - (u_m - u_o) \delta u_o \right] dx \\
+ \chi' \int_{\Omega} \left[ \nabla \cdot (\delta u_m \hat{N} H) + \nabla \cdot (u_m \hat{N} \delta H) + \nabla \cdot (u_m H \delta \hat{N}) - \delta \dot{a} \right] dx \\
+ \delta \chi' \int_{\Omega} \left( \nabla \cdot u_m \hat{N} H - \dot{a} \right) dx
\]
Gradients explained

Optimization requires *gradients*

Identification of terms in variation

\[
\delta I = \int_{\Omega_e} \left[ (u_m - u_o) \delta u_m - (u_m - u_o) \delta u_o \right] dx
\]

\[
+ \lambda' \int_{\Omega} \left[ \nabla \cdot (u_m \hat{N} H) + \nabla \cdot (u_m \hat{N} \delta H) + \nabla \cdot (u_m H \delta \hat{N}) - \delta \dot{a} \right] dx
\]

\[
+ \delta \lambda' \int_{\Omega} \left( \nabla \cdot u_m \hat{N} H - \dot{a} \right) dx
\]

*Adjoint RHS*  
*Adjoint LHS*  
*Forward Model*
Optimization cartoon
Directions can be simplified

She said, "Downhill"
Optimization cartoon
Downhill is good for the BFGS
North West region: speed results
Smoothed and interpolated with physics based PDE-constrained optimization
North East region: speed results
Smoothed and interpolated with physics based PDE-constrained optimization

![Final Vert. Avg. Speed (m/a)](image1)

![Observed Vert. Avg. Speed (m/a)](image2)
Central region: speed results
Smoothed and interpolated with physics based PDE-constrained optimization
Southern region: speed results
Smoothed and interpolated with physics based PDE-constrained optimization
Thickness (bed) results
Great interest in this, it conserves mass
Thickness (bed) results

More interesting to look at changes in thickness

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Apparent accumulation results

\[ \dot{a'} = \dot{a} - \frac{\partial H}{\partial t} \]

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Physics-based interpolation
Model intercomparison (MPAS)

Differences likely due to regularization

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Physics-based interpolation
Direction results

\( \hat{\mathbf{n}} = (n_x, n_y) \), \( n_y \) plotted here.
Conclusion

*Are the transients gone?*

- Transients in prognostic runs are lower.
- Speeds near terminus are not as “smooth” as the data show them to be.
- It’s not clear how good is good enough. Current RMSE $\sim 60$ m/a.
- The role of regularization and the objective function need to be explored.