A Stochastic Model of the Isopycnal Slope for use in Gent-McWilliam Parameterizations

Zofia Stanley

June 18, 2019
The density equation of state is a nonlinear function of temperature and salinity.
Averaging the nonlinear density function leads to errors.

\[
\rho(T, S) - \rho(\bar{T}, \bar{S})
\]
Jean-Michel Brankart proposed a stochastic parameterization of this error.

\[
\rho = \frac{1}{2n} \sum_{i=1}^{n} \rho(\bar{T} \pm \Delta T_i, \bar{S} \pm \Delta S_i, p_0(z))
\]
Brankart’s parameterization recovers the spatial structure of the density errors.
How does our approach differ from Brankart’s?

- Brankart:
  - Corrected average density through sampling
  - Used in hydrostatic equation to compute pressure gradient force

- Us:
  - Different correction
  - Will use in Gent-McWilliams parameterization
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We Taylor expand to find a correction to density.

\[
\rho(\bar{T} + \Delta T, \bar{S} + \Delta S) \approx \rho(\bar{T}, \bar{S}) + \Delta T \partial_T \rho(\bar{T}, \bar{S}) + \Delta S \partial_S \rho(\bar{T}, \bar{S}) \\
+ \frac{(\Delta T)^2}{2} \partial^2_T \rho(\bar{T}, \bar{S}) + (\Delta T \Delta S) \partial_T \partial_S \rho(\bar{T}, \bar{S}) \\
+ \frac{(\Delta S)^2}{2} \partial^2_S \rho(\bar{T}, \bar{S})
\]
Averaging over all possible fluctuations eliminates mean zero terms.

\[
\overline{\rho(T, S)} \approx \rho(\bar{T}, \bar{S}) + \Delta T \partial_T \rho(\bar{T}, \bar{S}) + \Delta S \partial_S \rho(\bar{T}, \bar{S}) \\
+ \frac{\sigma_T^2}{2} \partial_T^2 \rho(\bar{T}, \bar{S}) + \sigma_{TS} \partial_T \partial_S \rho(\bar{T}, \bar{S}) \\
+ \frac{\sigma_S^2}{2} \partial_S^2 \rho(\bar{T}, \bar{S})
\]
It suffices to use only the temperature term as our correction.

\[
\rho(T, S) \approx \rho(\bar{T}, \bar{S}) + \left( \frac{\sigma_T^2}{2} \right) \frac{\partial^2}{\partial T^2} \rho(\bar{T}, \bar{S})
\]
Our correction is good!
We need to model $\sigma_T^2$.

\[
\rho(T, S) \approx \rho(\bar{T}, \bar{S}) + \left( \frac{\sigma_T^2}{2} \right) \frac{\partial^2}{\partial T^2} \rho(\bar{T}, \bar{S})
\]
We model the variance as a length scale times the gradient of temperature.

\[ \sigma_T^2 \propto |\delta \nabla \bar{T}|^2 \]
Our mean model for the variance of temperature appears to fit the data well.
Our correction is still good with modeled $\sigma^2_T$. 

\[
\frac{\rho(T, S) - \rho(\bar{T}, \bar{S})}{0.14 \, |\nabla \delta x|^2 \frac{\partial^2}{\partial T^2} \rho(T, S)}
\]
We are working to model the residuals stochastically.
Where are we now?

- We are modeling the variance of temperature

\[ \sigma_T^2 \propto |\delta \nabla \bar{T}|^2 \]

So that we can model the error in density

\[ \rho(T, S) - \rho(\bar{T}, \bar{S}) \approx (\sigma_T^2)^2 \frac{\partial^2}{\partial T^2} \rho(\bar{T}, \bar{S}) \]

For use in the Gent-McWilliams parameterization of eddy velocity.

\[ u^* = \left( \kappa \nabla \rho \rho_z \right)_z \]
Where are we now?

- We are modeling the variance of temperature

\[ \sigma_T^2 \propto |\delta x \nabla \bar{T}|^2 \]

- So that we can model the error in density

\[ \bar{\rho}(T, S) - \rho(\bar{T}, \bar{S}) \approx \left( \frac{\sigma_T^2}{2} \right) \frac{\partial^2}{\partial_T^2} \rho(\bar{T}, \bar{S}) \]
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- For use in the Gent-McWilliams parameterization of eddy velocity.

\[ u^* = \left( \kappa \frac{\nabla \rho}{\rho z} \right)_z \]
Collaborators Alistair Adcroft, Scott Bachman, Fred Castruccio, Ian Grooms, William Kleiber

Support from the National Science Foundation

Reference Jean-Michel Brankart, *Impact of uncertainties in the horizontal density gradient upon low resolution global ocean modelling.*