Preconditioning Techniques Based on Domain Decomposition Methods

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Domain decomposition methods:
the process of subdividing the solution of a large system into smaller subproblems whose solutions can be used to produce a preconditioner for the system of equations that results from discretizing the PDE on the entire domain.
Idea of Domain Decomposition Methods

- Decompose the domain $\Omega$ into overlapping or non-overlapping subdomains.
- Assign one or several subdomains to each processor of parallel machine.

In each iteration:
- In each subdomain, solve small local subproblems.
- In addition, solve one small global problem.
Motivation

Conventional methods

- we usually need additional information, e.g., coarse coordinate information.
- we need quite regular meshes.
- it is hard to apply for irregular subdomains.
Alternative Approach

Generalized Dryja, Smith, Widlund (GDSW) coarse space technique

- this technique is based on energy minimizing discrete harmonic extensions.
- it has been applied to many applications
  - almost incompressible elasticity (Dohrmann, Widlund)
  - Reissner-Mindlin plates (Lee)
  - Raviart-Thomas vector fields (Oh)
Alternative Approach

Advantage

- the method can be implemented in an algebraic manner - we do not need any coarse discretization.
- it works well for irregular subdomains and unstructured meshes.
- it has well-established theoretical results, e.g., upper bounds of condition number.
A vector $u^{(i)} := [u^{(i)}_I u^{(i)}_\Gamma]^T$ is said to be discrete harmonic on $\Omega_i$ if
\[ A^{(i)}_I u^{(i)}_I + A^{(i)}_{i\Gamma} u^{(i)}_\Gamma = 0. \]

$u^{(i)}$ is completely defined by $u^{(i)}_\Gamma$.

The discrete harmonic extension has the minimal energy property.

\[ a(u, u) = \min_{v|\Gamma=u_\Gamma} a(v, v) \]
Coarse Component

Interface(Γ) : Vertex + Edge
Coarse Component

- $R_0$: restriction to coarse space
  - We choose one coarse edge or vertex and give 1 to the nodes on the edge or vertex.
  - We assign 0 to other nodes on the interface.
  - We use the discrete harmonic extension for interior parts.

- $A_0 = R_0 A R_0^T$

We note that this coarse component can be implemented in an algebraic manner. We do not need any coarse discretizations.
Additive Schwarz Preconditioner

Additive Schwarz Method for SPD systems

\[ P^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i \]

- \( A_0 \): coarse matrix (restriction to the coarse space)
- \( A_i \): local matrix (restriction to overlapping subdomain \( \Omega_i' \))
- \( R_0 \): restriction to coarse space
- \( R_i \): restriction to overlapping subdomain \( \Omega_i' \)
Restricted Additive Schwarz Perconditioner

Restricted Additive Schwarz Method for indefinite or nonsymmetric systems

\[ P^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} \tilde{R}_i^T A_i^{-1} R_i \]

- \( A_0 \): coarse matrix (restriction to the coarse space)
- \( A_i \): local matrix (restriction to extended subdomain \( \Omega_i' \))
- \( R_0 \): restriction to coarse space
- \( R_i \): restriction to overlapping subdomain \( \Omega_i' \)
- \( \tilde{R}_i \): restriction to subdomain \( \Omega_i \)
Numerical Experiments

5km Greenland Ice-Sheet
1 subdomain per each processor, preconditioned GMRES
local solver : Amesos KLU
coarse solver : Amesos KLU

Table: iteration counts

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<th># of processors</th>
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