Glacial FOSLS

New FOSLS Formulation of Nonlinear Stokes Flow for Glaciers

Jeffery Allen\textsuperscript{1}  Tom Manteuffel\textsuperscript{1}  Harihar Rajaram\textsuperscript{2}

\textsuperscript{1}University of Colorado Boulder
Department of Applied Mathematics

\textsuperscript{2}University of Colorado Boulder
Department of Civil, Environmental, and Architectural Engineering

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Outline:

1. Stokes for Glaciers
2. 2D Gravity Driven Glacier
3. Numerical Results
1. Stokes for Glaciers

2. 2D Gravity Driven Glacier

3. Numerical Results
Stokes for Glaciers

Viscosity Form

Continuity Equation:

\[ \nabla \cdot \mathbf{u} = 0 \]

Momentum Equation:

\[ 0 = \nabla \cdot \frac{1}{2} \mu \left( \nabla \mathbf{u} + \left( \nabla \mathbf{u} \right)^T \right) - \nabla p + \rho g, \]

Viscosity

\[ \mu = \frac{1}{2} A(T)^{-\frac{1}{3}} \dot{\varepsilon}_e^{-\frac{2}{3}}, \]

\[ \dot{\varepsilon}_e = \left| \left| \varepsilon_x \right| \right| F, \]

\[ \dot{\varepsilon} = \frac{1}{2} \left( \nabla \mathbf{u} + \left( \nabla \mathbf{u} \right)^T \right). \]
Stokes for Glaciers

FOSLS-ification

Rewrite as a First Order System

**Definition**

\[
\overrightarrow{U} = \nabla \overrightarrow{u} = \begin{bmatrix}
\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\
\frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\
\frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z}
\end{bmatrix} = \begin{bmatrix}
U_{11} & U_{21} & U_{31} \\
U_{12} & U_{22} & U_{32} \\
U_{13} & U_{23} & U_{33}
\end{bmatrix}
\]

**$H^1$ Elliptic system**

\[
\nabla \cdot \overrightarrow{u} = 0 \quad \text{(Continuity)}
\]

\[
\overrightarrow{U} = \nabla \overrightarrow{u} \quad \text{(Definition)}
\]

\[
\nabla \cdot \frac{1}{2} \mu (\overrightarrow{U} + \overrightarrow{U}^T) - \nabla p = -\rho g \quad \text{(Momentum)}
\]

\[
\nabla \times \overrightarrow{U} = 0 \quad \text{(Freebie)}
\]

\[
\text{Trace}(\overrightarrow{U}) = 0 \quad \text{(Enforced by setting $U_{11} = -U_{22}$)}
\]
The biggest problem with this formulation comes in the definition for viscosity.

\[
\mu = \frac{1}{2} A(T)^{-\frac{1}{3}} \dot{\varepsilon}_e^{-\frac{2}{3}}
\]

\[
\dot{\varepsilon}_e = ||\ddot{\varepsilon}||_F
\]

\[
\ddot{\varepsilon} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)
\]

The viscosity is near infinite where the glacier experiences small deformations. This is usually overcome by using a small constant in the effective strain rate.
Stokes for Glaciers
Fluidity Form

Definition

\[
\hat{U} = \hat{\varepsilon}^{-\frac{2}{3}} U
\]

\[
\hat{\varepsilon} = \frac{1}{2} \left( \hat{U} + \hat{U}^T \right)
\]

\[
\hat{\varepsilon}_e = \|\hat{\varepsilon}\|_F
\]

\[
\phi = \hat{\varepsilon}_e^2
\]

Notice that

\[
\phi = \hat{\varepsilon}_e^2 = \|\hat{\varepsilon}\|_F^2 = \|\hat{\varepsilon}^{-\frac{2}{3}} \hat{\varepsilon}\|_F^2 = \hat{\varepsilon}_e^{-\frac{4}{3}} \|\hat{\varepsilon}\|_F^2 = \hat{\varepsilon}_e^{-\frac{4}{3}} \hat{\varepsilon}_e^2 = \hat{\varepsilon}_e^\frac{2}{3}
\]

\[
\phi = \frac{1}{4} \left( (2\hat{U}_{11})^2 + 2(\hat{U}_{12} + \hat{U}_{21})^2 + (-2\hat{U}_{11})^2 \right)
\]
Fluidity FOSLS Equations

\[ \phi = 2\hat{U}_{11}^2 + \frac{1}{2}\hat{U}_{12}^2 + \frac{1}{2}\hat{U}_{21}^2 + \hat{U}_{21}\hat{U}_{21} \]

\[ \nabla \cdot u = 0 \]

\[ \phi \hat{U} = \nabla u \]

\[ \nabla \cdot \frac{1}{2} A(T)^{-\frac{1}{3}} \left( \hat{U} + \hat{U}^T \right) - \nabla p = -\rho g \]

\[ \nabla \times \phi \hat{U} = 0 \]

\[ U_{11} = -U_{22} \]
When $\phi$ is small, the Div and Curl equations are not of the same scale.

**Scaled Curl Equation**

$$\frac{1}{\phi + c} \nabla \times \phi \hat{U} = 0$$

This is equivalent to:

**Log Form (via: Product Rule)**

$$\nabla \times \hat{U} - (\nabla \perp \log(\phi + c)) \cdot \hat{U} = 0$$

Log Form has unscaled Curl equation with lower order terms.
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Parameters

- Bed Slope \((\theta) = 0.05\)
- \(A(T) = 4 \times 10^{-24} \text{ Pa}^{-3}\text{s}^{-1}\)
- \(|g| = 9.81 \text{ m/s}^2\)
- \(\rho = 900 \text{ kg/m}^3\)
- \(H = 1000 \text{ m}\)
- \(L = 10000 \text{ m}\)
- \(n = 3\)
- Assume pressure is zero on the surface
where $\theta$ is the bed slope $g = |g|[0, -1]^T$
now \( \overrightarrow{g} = |g|[\sin(\theta), -\cos(\theta)]^T \)
For the Top boundary we want to impose a stress free condition

\[ \sigma \cdot n = 0 \]

\[ \sigma = \hat{U} + \hat{U}^T - pI \]

\[
\begin{bmatrix}
2\hat{U}_{11} - p & \hat{U}_{12} + \hat{U}_{21} \\
\hat{U}_{21} + \hat{U}_{12} & 2\hat{U}_{22} - p
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
\hat{U}_{12} + \hat{U}_{21} \\
2\hat{U}_{22} - p
\end{bmatrix} = 0 \]
For the Top boundary we want to impose a stress free condition

\[ \mathbf{\sigma} \cdot \mathbf{n} = 0 \]

\[ \mathbf{\sigma} = \hat{\mathbf{U}} + \hat{\mathbf{U}}^T - pI \]

\[
\begin{bmatrix}
2\hat{U}_{11} - p & \hat{U}_{12} + \hat{U}_{21} \\
\hat{U}_{21} + \hat{U}_{12} & 2\hat{U}_{22} - p
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix} =
\begin{bmatrix}
\hat{U}_{12} + \hat{U}_{21} \\
\hat{U}_{11}
\end{bmatrix} = 0
\]
2D Gravity Driven Glacier
Boundary Conditions: Bottom & Sides

Assume the glacier is frozen to the bed (No Slip)

\[ u = 0 \]

This also gives us

\[ U_{11} = 0 \quad U_{21} = 0 \]
\[ \hat{U}_{11} = 0 \quad \hat{U}_{21} = 0 \]

Finally, assume the periodic side boundaries.
Assume the glacier is frozen to the bed (No Slip)

\[ u = 0 \]

This also gives us

\[ U_{11} = 0 \quad U_{21} = 0 \]
\[ \hat{U}_{11} = 0 \quad \hat{U}_{21} = 0 \]

Finally, assume the periodic side boundaries.
Notice that

\[ u = [u_1, u_2]^T = [f(z'), 0]^T \]

Using this and the other assumptions, we can backtrack to find the exact solution:

**Solution**

\[
\begin{align*}
    u_1 &= A (\rho |g| \sin(\theta))^3 (H^4 - (H - z)^4), \\
    U_{12} &= 4A (\rho |g| \sin(\theta))^3 (H - z)^3, \\
    p &= \rho |g| \cos(\theta) (H - z), \\
    \phi &= 2A^{\frac{2}{3}} (\rho |g| \sin(\theta))^2 (H - z)^2, \\
    \hat{U}_{12} &= 2A^{\frac{1}{3}} (\rho |g| \sin(\theta))(H - z), \\
    u_2 &= \hat{U}_{11} = \hat{U}_{21} = \hat{U}_{22} = 0.
\end{align*}
\]
2D Gravity Driven Glacier

Exact Solution

Downhill Velocity Profile

Problematic part of Viscosity

\[ u \left( \frac{1}{m/s} \right) \]

\[ z' \left( \frac{m}{m} \right) \]

\[ \varepsilon^{-\frac{2}{3}} \]

Jeffery Allen

Glacial FOSLS

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Numerical Results
Solver (Fospack)

1. Discretize PDE
2. Newton Iteration
   \[ Ax = b \text{ (Linearized)} \]
3. Refinement (ACE/Uniform)
4. Repeat Steps 1-4

Fospack Solver Scheme
Numerical Results

Functional Reduction

The graph illustrates the comparison between Fluidity Formulation and Viscosity Formulation. The red line represents Fluidity Formulation, and the green line represents Viscosity Formulation. The dashed line denotes $O(h)$, indicating the order of magnitude.

The x-axis represents the number of elements, ranging from 160 to 40,960. The y-axis shows the logarithm of the norm of the error, denoted as $\log(\|L u_h - f\|_0)$, with values ranging from $-3.4$ to $-4.6$.

The points on the graph correspond to different element counts, with annotations for selected data points, such as $-4.6$ at 160 elements, $-4.4$ at 640 elements, and so on. This graphical representation helps in understanding the convergence rate and the effectiveness of the formulations as the number of elements increases.
Numerical Results

$L^2$ Reduction

Final Error:
Viscosity Formulation: $1.8 \times 10^{-3}$
Fluidity Formulation: $1.3 \times 10^{-4}$

(14 times larger)
Numerical Results

AMG Factor: Comparison

\[0.98^q = 0.58 \quad q \approx 27\]
Summary of the fluidity formulation’s numerical performance. L is the level of refinement. N is the number of Newton steps. Complexity lists the cycle complexity for each Newton step. WU is the total number of work units for that level. Functional refers to the nonlinear functional norm.

<table>
<thead>
<tr>
<th>Level</th>
<th>E</th>
<th>Nonzeros</th>
<th>N</th>
<th>Complexity</th>
<th>V-Cycles</th>
<th>WU</th>
<th>Functional</th>
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<td>160</td>
<td>52000</td>
<td>2</td>
<td>3.59, 3.79</td>
<td>8, 7</td>
<td>0.121</td>
<td>4.13 × 10^{-4}</td>
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<tr>
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<td>640</td>
<td>196000</td>
<td>1</td>
<td>4.10</td>
<td>5</td>
<td>0.170</td>
<td>2.10 × 10^{-4}</td>
</tr>
<tr>
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<td>2560</td>
<td>760480</td>
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<td>4.44</td>
<td>4</td>
<td>0.570</td>
<td>1.05 × 10^{-4}</td>
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<tr>
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<td>2995360</td>
<td>1</td>
<td>4.60</td>
<td>3</td>
<td>1.745</td>
<td>5.25 × 10^{-5}</td>
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<tr>
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<td>1</td>
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<td>2.63 × 10^{-5}</td>
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<td></td>
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</tr>
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</table>
Numerical Results

Functional: Uniform Vs. ACE

![Graph showing the comparison between Uniform Refinement and ACE Refinement]

- Element counts: 160, 439, 640, 1540, 2560, 5875, 10,240, 21,622, 40,960
- Logarithmic scale for error norm $\|L u_h - f\|$
Conclusions

1. Glaciers are modeled by Stokes with nonlinear viscosity.
2. Viscosity becomes nearly infinite when the glacier experiences small deformations.
3. Nonlinear FOSLS formulation captures the physical behavior.
4. The fluidity formulation yields better numerical performance.
Future Work

- Benchmark Problems (ISMIP)
- Inclusion of Energy Model
- Time Dependent Domain
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Special Thanks to Tom, Steve, and Hari