Update on Greenland ice-sheet initialization: Optimal control and Bayesian calibration approaches

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Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.
Goal: recover initial ice-sheet state and avoid fast initial transients.

How to prescribe ice-sheet mechanical equilibrium:

\[
\frac{\partial H}{\partial t} = -\text{div} (UH) + \tau_s,
\]

\[
U = \frac{1}{H} \int u \, dz.
\]

At equilibrium:

\[
\text{div} (UH) = \tau_s
\]

Boundary condition at ice-bedrock interface:

\[
(\sigma n + \beta u)_{\parallel} = 0 \quad \text{on} \quad \Gamma_{\beta}
\]

Bibliography*:

Arthern, Gudmundsson, J. Glaciology, 2010

Price, Payne, Howat and Smith, PNAS 2011


Morlighem et al. A mass conservation approach for mapping glacier ice thickness, 2013
Deterministic Inversion (w/ S. Price and G. Stadler)
Estimation of ice-sheet initial state

PDE constrained optimization problem

Minimize mismatch between:
- *computed divergence flux* and *measured SMB*
- *computed and measured surface velocity*
- *computed and reference thickness*

Fulfill (Constraint):
- *High order nonlinear Stokes equation*

Tune (Control Variables):
- *Basal friction*
- *Thickness*

Caveat:
- Temperature field is given

\[ J(\beta, H) = \frac{1}{2} \int_\Gamma \frac{1}{\sigma_s^2} |\text{div}(UH) - \tau_s|^2 \, ds \quad (J^{SMB}) \]
\[ + \frac{1}{2} \int_{\Gamma_{top}} \frac{1}{\sigma_v^2} |u - u^{obs}|^2 \, ds \quad (J^{vel}) \]
\[ + \frac{1}{2} \int_\Gamma \frac{1}{\sigma_H^2} |H - H^{obs}|^2 \, ds \quad (J^H) \]
\[ + R(\beta) + R(H). \]

Software Tools:
- Assembling: LifeV
- Linear solver: AztecOO & IfPack (Trilinos).
- Nonlinear solver: NOX (Trilinos).
- Gradient Based optimization (LBFGS): Rol.
Consider ISMIP-HOM like forward simulation:

1. Tune \( \beta \) by matching surf. velocity.
2. Tune \( \beta \) by matching surf. vel. and SMB
3. Tune \( \beta \) and thick. by matching surf. vel. and SMB

\[
J^1(\beta) = J^{vel} + R
\]
\[
J^2(\beta) = J^{vel} + J^{SMB} + R
\]
\[
J^3(\beta, H) = J^{vel} + J^{SMB} + J^H + R
\]

Add noise to results of forward simulation to get \( u^{obs}, \tau_s \) and \( H^{obs} \)

Invert. We consider three cases:
Deterministic Inversion  (w/ S. Price and G. Stadler)

Numerical results

Slab example. Optimization results using different merit functionals

\[
J^1(\beta) \quad J^2(\beta) \quad J^3(\beta, H)
\]

\[
\beta \quad [kPa \ yr/m]
\]

\[
SMB \quad [m/yr]
\]
Slab example. Optimization results using different merit functionals

\[ J^1(\beta) \quad J^2(\beta) \quad J^3(\beta, H) \]

Surf. velocity [m/yr]
RMS error of surface velocity measures [m/yr] (left) and bedrock topography [km] (right).
Deterministic Inversion (w/ S. Price and G. Stadler)

Greenland initialization

Left, Center: Estimated beta obtained using different cost functionals. Right: difference between the computed and reference thickness in [km].
Deterministic Inversion (w/ S. Price and G. Stadler)

Greenland initialization

Left, center: computed surface velocity obtained with different functionals. Right: reference velocity. Units: [m/yr]
Deterministic Inversion (M. Perego, S. Price and G. Stadler)

Greenland initialization

Left, center: Estimated divergence flow obtained using different functionals.
Right: reference SMB.

SMB, [m/yr]
Bayesian Inversion
(w/ M. Eldred, J. Jakeman, I Kalashnikova, A. Salinger, L. Swiler)

Reduction of parameter space dimension

Difficulty in UQ approach: “Curse of dimensionality”. The parameter space has $O(30,000)$ parameters (or more).

- Reduce the dimension of the parameter space.

Method of choice: Karhunen-Loeve Expansion (KLE).
In our experiment, we reduce the dimension of parameter space to 5.

1. Assume analytic covariance kernel $C(r_1, r_2) = \exp\left(-\frac{|r_1 - r_2|^2}{L^2}\right)$.

2. Perform eigenvalue decomposition of $C$.

3. Take the mean $\bar{\beta}$ to be the deterministic solution and expand $\beta$ in basis of eigenvector $\{\phi_k\}$ of $C$, with random variables $\{\xi_k\}$

$$\beta(\omega) = \bar{\beta} + \sum_{k=1}^{K} \sqrt{\lambda_k} \phi_k \xi_k(\omega)$$

*Expansion done on log($\beta$) to avoid negative values for $\beta$.

Development(?): parameter reduction based on physical knowledge.
(e.g. include basal hydrology model)
Bayesian Inversion  
(w/ M. Eldred, J. Jakeman, I Kalashnikova, A. Salinger, L. Swiler)  

Reduction of parameter space dimension: Greenland modes

- 5 KLE modes capture 95% of covariance energy  
(parallel C++/Trilinos code Anasazi).

Only spatial correlation has been considered.

*Development(?):* Build modes using information from the model  
(e.g. using family of deterministic basal friction coefficients).
Bayesian Inversion  
(w/ M. Eldred, J. Jakeman, I. Kalashnikova, A. Salinger, L. Swiler)

Compute model surrogate and invert

• Mismatch \( \text{(ALBANY)}: \mathcal{J}(\beta) = \frac{1}{2} \int_{\Gamma} \frac{1}{\sigma_s^2} |\text{div}(UH) - SMB|^2 \, ds. \)

• **Build Surrogate Model.** Polynomial chaos expansion (PCE) was formed for the mismatch over random variables using uniform prior distributions. **DAKOTA.**

• **Inversion/Calibration.** Markov Chain Monte Carlo (MCMC) was performed on the PCE with 100K samples **QUESO.**

*Development(?):* use simple physical model (e.g. \( L1L2 \) or \( SIA \)) as the surrogate model.
Posterior distributions for the 5 KLE coefficients:

MAP solution: $\xi = (-0.16, -0.08, 0, 0, 0)$
Deterministic beta [kPa yr/m]

Bayesian beta [kPa yr/m]

Bayesian Inversion
(w/ M. Eldred, J. Jakeman, I Kalashnikova, A. Salinger, L. Swiler)

Numerical Results