Modeling the Fracture of Ice Sheets on Parallel Computers

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Disintegrating Ice Shelves
Since the mid-1980s

2000 km\(^2\) of Larsen Ice Shelf disintegrated in 2 days.
Source: T. Scambos

Require >10,000 years to form
Disintegrate in weeks

Slide from Dr. Bindschadler presentation, DOE workshop Sep. 2009
FEA Model Description

Geometry
- Strip model with degrading stiffness towards tip

Temperature dependent properties
- Elastic
- Thermal

Boundary Conditions
- Fixed edges
- Elastic foundation

Loads
- Body gravity Load
- Temperature Increase

Analysis Steps
- Static, General with body load
- Coupled Temp-Disp – steady state
### Basic Equations

#### Static Force Equilibrium:
\[ \sigma_{i,j,j} + f_i = 0 \]

#### Constitutive Law:
\[ \sigma^{el}_{ij} = C_i{ij}_k \varepsilon_{ij} = D^{el}(\theta) \cdot \varepsilon \]

#### Deformation:
- Elastic: \[ \varepsilon^{el}_{ij} = \frac{1}{2} (u_{j,i} + u_{i,j}) \]
- Expansion: \[ \varepsilon^{th}_{ij} = \alpha \Delta \theta \delta_{ij} \]
- Total-Strain: \[ \varepsilon_{ij} = \varepsilon^{el}_{ij} + \varepsilon^{th}_{ij} \]

#### Fourier Law:
\[ q = -\kappa \nabla \theta \]

#### Thermal Equilibrium:
\[ -\kappa \nabla^2 \theta + c_p \rho \dot{\theta} = q \]

#### Material Properties:
- \( \kappa(\theta) \): Material conductivity
- \( c_p(\theta) \): Specific heat capacity
- \( \rho(\theta) \): density

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Note: Since the time scales for (a) temperature change (which affects both the elastic and thermal properties) and (b) thermal equilibrium are vastly different, these two sets of conditions are only weakly coupled and can be solved independently.
# Ice – Physical, Thermal and Mechanical Properties

## Temperature, Density, Thermal Conductivity, and Specific Heat

<table>
<thead>
<tr>
<th>Temperature (T °C)</th>
<th>Density (ρ kg/m^3)</th>
<th>Thermal Conductivity (K W/m.K)</th>
<th>Specific Heat (Cp J/kg.K)</th>
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<td>1.39E-03</td>
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## Elastic Modulus and Temperature

<table>
<thead>
<tr>
<th>Elastic Modulus (E GPa)</th>
<th>Temperature (Temp °C)</th>
</tr>
</thead>
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<tr>
<td>6.0</td>
<td>0</td>
</tr>
<tr>
<td>7.0</td>
<td>-10</td>
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<td>7.5</td>
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<td>-30</td>
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<td>-50</td>
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<tr>
<td>10.0</td>
<td>-60</td>
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<tr>
<td>11.0</td>
<td>-80</td>
</tr>
<tr>
<td>12.0</td>
<td>-100</td>
</tr>
</tbody>
</table>

The elastic modulus properties have a wide range of variability and is generally strongly orthotropic. In the preliminary analysis, it was assumed to be linearly isotropic with softening at higher temperature.

**Thermal Expansion**

\[
\alpha = 5.1E-05 /\text{deg. C}
\]
Temperature Rise

Reference:
http://data.giss.nasa.gov/gistemp/graphs/
Preliminary Simulation
How XFEM works

Displacements written as

\[ u^h(x) = \sum_{I=1}^{n} N_I(x) u_I + \sum_{I=1}^{n_J} N_I(x) H(x) a_I + \sum_{I=1}^{n_T} \left[ N_I(x) \sum_{j=1}^{4} F_j(x) b_{jI} \right] \]

Heaviside jump-enrichment

\[ H(X) = \begin{cases} 
1 & \text{above } \Gamma_c^+ \\
0 & \text{bellow } \Gamma_c^- 
\end{cases} \]

Levelset Method

Belytschko et. al., IJNME [1999]
\[ u(x) = \sum_i \phi_i(x)u_i + \sum_i \phi_i(x)H(x)z_i + \ldots, \quad H(x) = \begin{cases} 1 & \varphi \geq 0 \\ 0 & \varphi < 0 \end{cases}, \quad \Gamma = \{(x, y) | \varphi(x, y) = 0\} \]

- Very large linear system
- Variable block size matrix with peculiar basis functions

Alternative view of matrix:

\[ A = \begin{pmatrix} A_{ss} & A_{sw} \\ A_{ws} & A_{ww} \end{pmatrix} \]

⇒ Need fast highly parallel solver!
Algebraic Multigrid Background

- Construct graph & coarsen
- Determine $P_i$'s sparsity pattern
- Determine $P_i$'s coefs
- Project: $A_i = (P_i)^T A_{i+1} P_i$

Solve $A_3 u_3 = f_3$

Smooth $A_3 u_3 = f_3$, $f_2 \leftarrow (P_2)^T r_3$

Smooth $A_2 u_2 = f_2$, $f_1 \leftarrow (P_1)^T r_2$

Solve $A_1 u_1 = f_1$ directly

$u_3 \leftarrow u_3 + P_2 u_2$

$u_2 \leftarrow u_2 + P_1 u_1$

Brandt, McCormick, Ruge, Stubën
Basic Idea

**AMG & Energy Minimization**

*Tradeoffs:*

- **flexibility**
  - any coarsening
  - any sparsity pattern
  - constraints
  - **important modes** requiring accurate interpolation

- **robustness**

**Constraints:**
accurate interpolation of certain modes, e.g. constant energy

*Graph showing high and low energy levels.*
**XFEM philosophy**

- standard grid
- standard basis functions
- weird basis functions

**Our AMG philosophy**

- standard coarsening
- standard prolongator columns
- weird prolongator columns

\[
A = \begin{pmatrix}
A_{ss} & A_{sw} \\
A_{ws} & A_{ww}
\end{pmatrix}
\]

\[
P = \begin{pmatrix}
P_{ss} & P_{sw} \\
0 & P_{ww}
\end{pmatrix}
\]

- \(P_{ss} \leftarrow \text{Standard AMG}(A_{ss})\)
- \(P_{ww} \leftarrow I\) (currently not coarsening special functions)
- \(P_{sw}\) use standard energy minimization
  - determine sparsity pattern
  - special interpolation constraints

Standard FEM requires special meshes to model flaws.
**Energy Minimization**

- Boils down to repeated iteration on
  \[ A_{ss} P_{sw} = -A_{sw} \]
  followed by sparsity restriction & enforcing mode constraints
- Without sparsity limits
  \[ \Rightarrow P_{sw} = -(A_{ss})^{-1} A_{sw} \]
  (Schur complements)

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**Sparsity Pattern**

\[ N = p(A_{ss}) A_{sw} \]
for experiments \( p(A_{ss}) = A_{ss} \)

---

**Special constraints**

- 3 zero energy modes corresponding to discontinuity captured by XFEM
Special Constraints

Standard AMG
Special Constraints

Only need to add special constraints near crack

Standard AMG & XFEM
### Results for 1 crack

<table>
<thead>
<tr>
<th>n x n grid</th>
<th>AMG on ( A_{ss} )</th>
<th>( Z=0 ) AMG</th>
<th>( Z \neq 0 ) AMG</th>
<th>( Z \neq 0 ) AMG + constraints</th>
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<tr>
<td>6</td>
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<td>162</td>
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<td>29</td>
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Crack crosses domain, aligned coarsening

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Partial domain crossing, coarsening aligned

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<td>43</td>
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<td>12</td>
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Crack crosses domain, nonaligned coarsening

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<th>n x n grid</th>
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<td>162</td>
<td>10</td>
<td>50</td>
<td>27</td>
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</table>

Partial crossing, nonaligned coarsening

\[
P = \begin{pmatrix} P_{ss} & Z \\ 0 & I \end{pmatrix}
\]
Conclusions

Initial studies to explain the collapse (disintegration) of an ice shelf with simple models

Ice is a complicated material

- Further research is needed to characterize the material behavior

The model may provide the crack initiation for the fracture modeling

Algebraic multigrid can mirror XFEM idea!

- Special prolongator columns to capture discontinuities

Promising results, but …

- Still need to resolve special constraints near crack tip & determine special constraints for crack network