Textbook multigrid efficiency for hydrostatic ice flow

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Hydrostatic equations

- Valid in the limit $w_x \ll u_z$, independent of basal friction
- Eliminate $p$ and $w$ by incompressibility:
  3D elliptic system for $\mathbf{u} = (u, v)$

\[-\nabla \cdot \left[ \eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix} \right] + \rho g \nabla s = 0\]

\[\eta(\gamma) = \frac{B}{2} (\epsilon^2 + \gamma)^{\frac{1-n}{2n}}, \quad n \approx 3\]

\[\gamma = u_x^2 + v_y^2 + u_x v_y + \frac{1}{4} (u_y + v_x)^2 + \frac{1}{4} u_z^2 + \frac{1}{4} v_z^2\]

and slip boundary $\sigma \cdot \mathbf{n} = \beta^2 \mathbf{u}$ where

\[\beta^2(\gamma_b) = \beta_0^2 (\epsilon_b^2 + \gamma_b)^{\frac{m-1}{2}}, \quad 0 < m \leq 1\]

\[\gamma_b = \frac{1}{2} (u^2 + v^2)\]

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1C. Schoof and R. Hindmarsh, *Thin-film flows with wall slip: an asymptotic analysis of higher order glacier flow models*, 2010
Hydrostatic solver

- 3D elliptic system for \((u, v)\)
  
  \[-\nabla \cdot \left[ \eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix} \right] + \rho g \nabla s = 0\]

- strong anisotropy and viscosity variation, shear important

- current solvers
  - Picard iteration
  - serial linear solve or 1-level ASM, high iteration counts
  - slow convergence with nonlinear sliding

- `petsc/src/snes/examples/tutorials/ex48.c`
  - conforming \(Q_1\) finite element discretization, no \(\sigma\)-transform
  - multigrid Newton-Krylov
  - smoother respects strong coupling in vertical: rapid coarsening
  - globalized by grid sequencing
  - quadratic convergence including slip conditions
  - robust to discontinuous sliding, high aspect ratio
  - high throughput: SSE2 integration, blocked matrix formats
Convergence with grid sequencing

- colored points are nonlinear residuals
- black × marks are unpreconditioned linear residuals
Picard with ASM overlap $1/\text{ICC}(1)$, 8 subdomains

- $n = 3$, $m = 1/3$, $\frac{\eta_{\text{max}}}{\eta_{\text{min}}} \approx 380$
- Geometry of ISMIP-HOM test A, “dimpled sombrero” sliding
- Slow nonlinear convergence, over 100 iterations for rtol $10^{-2}$
Picard with multigrid preconditioning, 512 subdomains

- slow nonlinear convergence
- very fast linear solves (note rtol is now $10^{-5}$)
Picard without grid sequencing

- slow initial convergence
Picard for rheology, Newton-linearized sliding

- cannot Newton-linearize only sliding, step is not descent direction
Switch to Newton

- much faster nonlinear convergence
- linear systems slightly more difficult
- initial nonlinear convergence is slow
Nonlinear convergence

- ASM/ICC(0) smoothers, isotropic coarsening
- 512 subdomains on fine levels
Avoid oversolving

Luis Chacon’s variant of Eisenstat-Walker

almost equivalent nonlinear convergence
Linear solve performance

- Newton, ICC(0), serial
- Picard, ASM overlap 1/ICC(1), 8 subdomains
- Newton, V-cycle multigrid, 512 subdomains

![Graph showing linear solve performance](image-url)
Aggressive coarsening

- Coarsest level is responsible for providing global coupling
- Subdomains very small on coarse levels: bottleneck is latency
- Approaches to anisotropy
  - Semi-coarsening ($z$-only): many intermediate levels, expensive
  - Line smoothers: allow rapid coarsening
- Order unknowns so that ICC(0) performs exact column solves
  - Allows isotropic coarsening at moderate aspect ratios
  - Allows semicoarsening by factors $\geq 8$
Nonlinear convergence: $n = 3$, $m = 1$, $\eta_{\text{max}}/\eta_{\text{min}} = 3800$

- element aspect ratio 640
- discontinuous sliding, strong shear zone (0.5 m/a to 46 km/a)
- rapid semi-coarsening, then isotropic quasi-2D coarsening
Same problem, rtol $10^{-2}$

almost equivalent convergence
Breakdown of incomplete factorization

- Jacobian is symmetric positive definite
- full Cholesky always has positive pivots
- incomplete Cholesky can generate negative pivots
- increasing fill can make the problem worse
- standard practice is to “shift” diagonal
  - may still be an adequate local smoother
  - destroys nonlocal properties of preconditioner
  - increasing overlap often makes it worse
- additive Schwarz with incomplete Cholesky is unreliable
- additive Schwarz with direct solves is reliable but slow
Conclusions

- Newton is almost 10 times faster than Picard
- deficiencies of serial incomplete factorization is more evident with Newton than Picard
- incomplete factorization is poor with 1-level domain decomposition
- multigrid preconditioning is \( \sim 50 \) times faster at \( \sim 1M \) nodes
- grid sequencing is important for globalization
- preconditioner can be lagged with true Jacobian applied matrix-free